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$N = 2$ supergravity in $d = 4, 5, 6$ and its matter couplings

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Abstract

An overview of $N = 2$ theories, with 8 real supercharges, in 4,5 and 6 dimensions is given. The construction of the theories by superconformal methods is explained. Special geometry is obtained and characterized. The relation between the theories in those dimensions is discussed. This leads also to the concept of very special geometry, and quaternionic-Kähler manifolds. These structures are explained.

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Contents

1 Introduction

These lectures will treat theories with 8 supercharges. Why are the theories with 8 supersymmetries so interesting? The maximal supergravities[1] contain 32 supersymmetries. These are the $N = 8$ theories in 4 dimensions, and exist in spaces of Lorentzian signature with at most 11 dimensions, i.e. (10,1) spacetime dimensions. If one allows more time directions, 32 supersymmetries is possible in 12 dimensions with (10,2) or (6,6) signature. However, these theories allow no matter multiplets[2]. For the geometry, determined by the kinetic terms of the scalars, this means that the manifold is fixed once the dimension is given. For all theories with 32 supersymmetries this is a symmetric space.

Matter multiplets are possible if one limits the number of supersymmetries to 16 (thus $N = 4$ in 4 dimensions). These theories exist up to 10 dimensions with Lorentzian signature. In this case, the geometry is fixed to a particular coset geometry once one gives the number of matter multiplets that are coupled to supergravity.

The situation becomes more interesting if the number of supersymmetries is 8. Now there are functions, which can be varied continuously, that determine the geometry. This makes the geometries much more interesting. Of course, if one further restricts to 4 supersymmetries, more geometries would be possible. In 4 dimensions, e.g., general Kähler manifolds appear. For 8 supersymmetries, these are restricted to 'special Kähler manifolds', determined by a holomorphic prepotential [1]. However, this restriction makes the class of manifolds very interesting and manageable. The holomorphicity is a useful ingredient, and was e.g. essential to allow the solution of the theory in the Seiberg–Witten model [2, 3]. The theories with 8 supersymmetries are thus the maximally supersymmetric that are not completely determined by the number of fields in the model, but allow arbitrary functions in their definition, i.e. continuous deformations of the metric of the manifolds.

I will consider mostly $N = 2$ in 4 dimensions, but also comment on $N = 2$ in 5 dimensions and in 6 dimensions. Concerning terminology, $N = 2$ in 6 dimensions is what is also called $(1, 0)$ in 6 dimensions. Indeed, in 6 dimensions one can have chiral and antichiral real generators, and the classification can thus be done by giving multiples of 8 real chiral or 8 real antichiral spinors. 8 chiral spinors is the minimal spinor of $Spin(5, 1)$. Also for $N = 2$ in 5 dimensions, the minimal spinor of $Spin(4, 1)$ has 8 real components. I denote these theories still as $N = 2$ because for practical work we always use doublets of spinors. Note also that for other signatures (2 or 3 time directions) one can impose Majorana conditions such that only 4 of them survive ($N = 1$). But for Minkowski signature, one can only have (symplectic) Majorana conditions, which need doublets of 4-component spinors. The basic properties of the spinors are repeated in appendix A.2. Thus, we will always have an SU(2) automorphism group of the spinors (and for 4 dimensions the automorphism group has an extra $U(1)$).

In the second superstring revolution, the models with special geometry were considered

[1]The restriction is due to interacting field theory descriptions, which e.g. in 4 dimensions does not allow fields with spin larger than 2.

[2]We distinguish the multiplet that contains the graviton and gravitini, and is determined by specifying the dimension and the number of supersymmetries, and other multiplets, which we call 'matter multiplets'.

first, due to this restrictive nature, while still allowing functions that can vary continuously. The concept of duality became very important at that time. With Bernard de Wit, we had considered these dualities in $N = 2$, $d = 4$ earlier, and this concept had a natural formulation in the context of superconformal tensor calculus.

Superconformal tensor calculus is another main topic of these lectures. It has been the basis of the first constructions of general matter couplings. It was initiated in $N = 1$ due to the work of S. Ferrara, M. Kaku, P.K. Townsend and P. van Nieuwenhuizen [4, 5, 6]. The extra symmetries of the superconformal group give an advantage over the direct super-Poincaré approach in that many aspects of the theory get a clear structure. In fact, the natural vectors in which the dualities have to be formulated are the multiplets of the superconformal tensor calculus. In this approach the superconformal symmetry is used as a tool to obtain the theories that have super-Poincaré symmetry. All the super-Poincaré theories are constructed as explicitly broken superconformal theories. Lately the conformal symmetry has gained importance. There is of course its interest in AdS/CFT correspondence, although in that context one mostly considers rigid conformal symmetry. But it may also be interesting to consider which parts of the supergravity theory are explicitly determined by the breaking of the superconformal invariance, and which part is generically determined by the superconformal structure. Recently this was useful in cosmology issues related to gravitino production in the early universe [7]. Thus the superconformal approach may be more interesting than just as a tool to obtain super-Poincaré results.

The conformal tensor calculus is not the only one to obtain the theories that we consider. For the superspace methods you may consider [8]) and for the harmonic superspace methods [9]. Furthermore there is the rheonomic approach that has been used for another formulation of the general $N = 2$ theories in four dimensions [10], and in five dimensions [11].

The plan of the lectures is as follows. Section 2 will introduce some basic concepts. That involves at the end general formulae for gauge theories. However, the gauging of the spacetime symmetries is not that straightforward, and one needs constraints. The gauging of the superconformal group is not a trivial issue and is dealt with in section 3 to obtain the so-called Weyl multiplet, the multiplet of gauge fields of the superconformal theory. I will give some practical tools to work with covariant derivatives.

XXX to be improved XXX Then I introduce matter (section 4). Matter multiplets are first introduced as representations the superconformal algebra, and then the construction of their actions is considered. Due to time limits only a few multiplets will be treated. But that is sufficient to discuss the structure of special geometry in section 6. This structure is also defined independently of its supergravity construction. It appears in moduli spaces of Calabi–Yau threefolds.

The matter couplings with $N = 2$ in 5 and 6 dimensions are of course very much related to those in four dimensions. In section 8, I discuss the differences and specific properties of the $N = 2$ theories in these dimensions. Very special geometry will then show up in relations between the scalar manifolds that these theories define. This is further clarified in relations between the homogeneous spaces that they allow. I will also give there an introduction to quaternionic spaces.

A large appendix is given with conventions and tricks for handling spinors and gamma matrices.

XXX superconformal tensor calculus, refer to previous reviews as thesis of M. Derix, S. Cucu, ...

2 Basic ingredients.

This section is meant to introduce some tools that are useful for the construction of superconformal gauge theory and multiplets. First I discuss the catalogue of supersymmetric theories with 8 supercharges and their multiplets. Then the superconformal groups are discussed. Finally, I repeat the general formulae for gauge theories.

2.1 Overview of supersymmetric theories with 8 real supercharges

It is clear that one can introduce 8 supercharges in 1 or 2 dimensions, where the elementary spinors have just one component. In 3 dimensions, gamma matrices are 2×2 matrices, and the theories with 8 supercharges are $N = 4$ theories where the spinors satisfy a reality condition (a Majorana condition). $d = 3$ will be mentioned shortly in section 8.3, but for the rest of the lectures, I will consider $d = 4, 5$ and 6. The latter is the maximal dimension for theories with 8 supercharges. In fact, one could first construct these theories and then derive several results for $d < 6$ from dimensional reduction. This programme has been started for the superconformal theories in [12, 13]. There are many aspects that can be treated at once for $d = 4, 5, 6$. The treatment of the Weyl multiplets is for a large part the same. Hypermultiplets (multiplets with scalars and spinors only) do not feel the difference of dimension. The difference indeed sits in the vectors and tensors, which under dimensional reduction decompose in several representations of the Lorentz algebra. Only a subset of the couplings of vectors multiplets in $d = 4$ and $d = 5$ can be obtained from dimensional reduction, as I will explicitly demonstrate in section 8.3. Therefore I will present the theories for each dimension separately.

In 4 dimensions, the supersymmetries are represented by Majorana spinors. But in practice, one can also use chiral spinors. I refer to the appendix A.2.1 for the notation and definitions of the properties of the spinors. A supersymmetry operation is represented as

$$\begin{aligned} &\delta(\epsilon) = \bar{\epsilon}^i Q_i + \bar{\epsilon}_i Q^i \,, \\ &\epsilon^i = \gamma_5 \epsilon^i \,, \qquad Q_i = \gamma_5 Q_i \,, \qquad \epsilon_i = -\gamma_5 \epsilon_i \,, \qquad Q^i = -\gamma_5 Q^i \,. \end{aligned} \tag{2.1}$$

Q is the supersymmetry operation that acts on the field that follows. E.g. if $\delta(\epsilon)X = \frac{1}{2}\bar{\epsilon}^i \Omega_i$, then $Q_i X = \frac{1}{2}\Omega_i$ and $Q^i X = 0$.

In 5 dimensions, one uses symplectic Majorana spinors. The reality rules of appendix A.2.2 now imply that we have to insert a factor i (see also the different meaning of the position of the indices i, as explained in appendix A.2.2):

$$\delta(\epsilon) = \mathrm{i}\bar{\epsilon}^i Q_i \,. \tag{2.2}$$

In 6 dimensions, symplectic Majorana–Weyl spinors can be used. In this case we have

$$\begin{aligned}
&\delta(\epsilon) = \bar{\epsilon}^i Q_i\,,\\
&\gamma_7\epsilon^i = \epsilon^i\,, \qquad \gamma_7 Q_i = -Q_i\,.
\end{aligned} \tag{2.3}$$

The properties of the spinors also imply how the translations appear in the anticommutator of two supersymmetries (the overall real factor is a matter of choice of normalization):

$$\begin{aligned}
d=4 \quad &: \quad \left\{Q^i_\alpha, Q_{\beta j}\right\} = -\tfrac{1}{2}\left(\gamma_a P_L\right)_{\alpha\beta}\delta^i_j P_a\\
d=5 \text{ and } d=6 \quad &: \quad \left\{Q^i_\alpha, Q^j_\beta\right\} = -\tfrac{1}{2}\varepsilon^{ij}\left(\gamma_a\right)_{\alpha\beta} P_a\,.
\end{aligned} \tag{2.4}$$

On the use of spinor indices, see appendix A.3.1. The normalization of the spinor generators is chosen to agree with existing literature in each case.

Exercise 2.1: Determine the parameter that will appear in the translation corresponding to (2.4). Thus, determine ξ^a in

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \xi^a(\epsilon_1, \epsilon_2) P_a\,. \tag{2.5}$$

The results are

$$\begin{aligned}
d=4 \quad &: \quad \xi^a(\epsilon_1,\epsilon_2) = \tfrac{1}{2}\left(\bar{\epsilon}_2^i\gamma^a\epsilon_{1i} + \bar{\epsilon}_{2i}\gamma^a\epsilon_1^i\right)\\
d=5 \text{ and } d=6 \quad &: \quad \xi^a(\epsilon_1,\epsilon_2) = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1\,.
\end{aligned} \tag{2.6}$$

2.2 Multiplets.

The classical reference for the classification of multiplets for extended supersymmetry is [14]. From that reference, I extract the on-shell massless representations given in table 1. In the table in [14] are given also different SU(2) assignments, but these are not unique (see e.g. below the linear multiplet as alternative to the hypermultiplet, so I omit these. To recognize the number of on-shell components for massless fields, one has to view them as representations of SO($d-2$). As such, the vectors have $d-2$ components. The gravitons are symmetric traceless tensors, and have thus $\frac{1}{2}(d-2)(d-1)-1 = d(d-3)/2$ components. The gravitini are γ - traceless spinors. As spinors have in each of these cases 2 components (for $d=6$ due to the chirality), this gives $(d-2)2-2 = 2(d-3)$ components (and a factor 2 from the SU(2)). One can check the boson–fermion balance for all the multiplets.

Remark that vector multiplets have complex scalars in 4 dimensions, real scalars in 5 dimensions and no scalars in 6 dimensions. The scalars in $d<6$ can thus be considered as the remainders of the extra components of the $d=6$ vectors.

For the hypermultiplets, one actually needs the double of the fields mentioned in the table to construct actions. Indeed the scalars are doublets, and can thus not be real. See also the lectures of P. West XXX in this respect. Then there are 4 real scalars in each multiplet and supersymmetry defined 3 complex structures, such that the scalars naturally combine in quaternions.

Table 1: *Massless on-shell representations. The fields are given a name as they will appear later. The representation content is indicated as its multiplicity in* $(\mathrm{SO}(d-2),\mathrm{SU}(2))$*, where the latter is also obtained from the* i*-type indices. For 6 dimensions, one can write* $\mathrm{SO}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)$*, and the corresponding decomposition is written. The non-symmetric representations are either chiral (chirality is indicated on the field) or they are antisymmetric tensors with self-dual (+) or anti-self-dual (−) field strengths. Of course, the two-form indicated here is not ± self dual, only its field strength is.*

$d=4$	$g_{(\mu\nu)}$	ψ^i_μ	V_μ		gravity
SO(2)	$(2,1)$	$(2,2)$	$(2,1)$		
	W_μ	Ω_i	X	X^*	vector multiplet
	$(2,1)$	$(2,2)$	$(1,1)$	$(1,1)$	
		ζ	A^i		hypermultiplet
		$(2,1)$	$(1,2)$		
$d=5$	$g_{(\mu\nu)}$	ψ^i_μ	A_μ		gravity
SO(3)	$(5,1)$	$(4,2)$	$(3,1)$		
	A_μ	λ^i	ϕ		vector multiplet
	$(3,1)$	$(2,2)$	$(1,1)$		
		ζ	A^i		hypermultiplet
		$(2,1)$	$(1,2)$		
$d=6$	$g_{(\mu\nu)}$	$\psi^i_{\mu L}$	$B^-_{\mu\nu}$		gravity
SO(4)	$(3,3;1)$	$(2,3;2)$	$(1,3;1)$		
	$B^+_{\mu\nu}$	ψ^i_R	σ		tensor multiplet
	$(3,1;1)$	$(2,1;2)$	$(1,1;1)$		
	W_μ	Ω^i_L			vector multiplet
	$(2,2;1)$	$(1,2;2)$			
		ζ_R	A^i		hypermultiplet
		$(2,1;1)$	$(1,1;2)$		

Remark that here the on-shell massless multiplets have been mentioned. The same multiplet may be represented by different off-shell multiplets. In 4 dimensions, physical scalars may be dualized to antisymmetric tensors. E.g. when one of the scalars of a vector multiplet is replaced by an antisymmetric tensor, we obtain the *vector–tensor multiplet* [15, 16, 17]

$$\text{vector–tensor multiplet: } V_\mu,\ \lambda^i,\ \phi,\ B_{\mu\nu}\,. \tag{2.7}$$

When one of the scalars of a hypermultiplet is replaced by an antisymmetric tensor, then this gives the so-called *linear multiplet* (the name is due to its relation with a superfield that has a linear constraint)

$$\text{linear multiplet: } \varphi^i,\ L^{(ij)},\ E_a\,. \tag{2.8}$$

I write the antisymmetric tensor here as a vector E_a that satisfies a constraint $\partial^a E_a = 0$, such that $E_a = \varepsilon_{abcd}\partial^b E^{cd}$. In this way, it can be generalized to $d = 5$ and $d = 6$ in a similar way. In these cases the E_a is the field strength of a 3-form or 4-form, respectively.

In 5 dimensions a vector is dual to a 2-index antisymmetric tensor. Therefore the vector multiplets can be dualized to antisymmetric tensor multiplets:

$$d = 5 \text{ antisymmetric tensor multiplet: } H_{\mu\nu},\ \lambda^i,\ \phi\,. \tag{2.9}$$

For non-Abelian multiplets, the two formulations are not equivalent. This has been investigated in detail in a series of papers of M. Günaydin and M. Zagermann [18, 19, 20]. The above overview is not exhaustive.

2.3 The strategy

Our aim is to study the transformation laws and actions for all these multiplets, coupled to supergravity. As mentioned in the introduction, there are several ways to accomplish this. The method that we use is the superconformal tensor calculus. That means that we use extra symmetry. We use as much symmetry as possible. That means that we will in intermediate steps use symmetries that will not be present in the final action. But its use has two advantages. It facilitates the construction of the theories, and secondly, it will clarify a lot of the structure of the theories.

The extra symmetry will be the superconformal symmetries. The motivation for this lies in the classical work of Coleman and Mandula [21], who proved that the largest spacetime group that is allowed without implying triviality of all scattering amplitudes is the conformal group. Although our motivation and use of the group is completely different, this gives an indication that the use of this group may be the most advantageous strategy. Over the years we got more convinced that indeed the use of conformal symmetries is a very useful and clarifying method. Analyzing the steps that are taken in local superspaces, we see that after using part of the superspace constraints, the remaining part that leads to more insight is equivalent to the structure that we use in the 'superconformal tensor calculus'.

What we have in mind can be illustrated first for pure gravity. We show how Poincaré supergravity is obtained after gauge fixing a conformal invariant action. The details of this example will come back in section 4.1. We now just give the general idea. The conformal invariant action for a scalar ϕ (in 4 dimensions) is[3]

$$\begin{aligned}\mathcal{L} &= \sqrt{g}\left[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)+\tfrac{1}{12}R\phi^2\right] ,\\ \delta\phi &= \Lambda_D\phi\,, \qquad \delta g_{\mu\nu}=-2\Lambda_D g_{\mu\nu}\,,\end{aligned} \tag{2.10}$$

where the second line gives the local dilatation symmetry that leaves this action invariant. Now, we can gauge fix this dilatation symmetry by choosing[4] the gauge

$$\phi=\sqrt{\frac{6}{\kappa}}\,. \tag{2.11}$$

This leads to the pure Poincaré action

$$\mathcal{L}=\tfrac{1}{2\kappa^2}\sqrt{g}R\,. \tag{2.12}$$

Pure Poincaré is in this way obtained from a conformal action of a scalar after gauge fixing. This scalar, which we will denote further as 'compensating scalar', thus has no physical modes. Note also that the mass scale of the Poincaré theory is introduced through the gauge fixing (2.11).

What we have seen is 1) the use of conformal symmetry, 2) construction of a conformal invariant action, 3) gauge fixing of superfluous symmetries.

In the remaining part of this chapter, we will make familiarize ourselves first with the conformal group, and first of all as a rigid symmetry. We will then take a look at possible superconformal groups, and repeat the basis rules for gauging symmetries. In section 3 we will learn how to gauge the superconformal group. The step 2) of the above overview involves the superconformal construction of multiplets and their action. This will be the subject of section 4. The step 3) will be taken in section 5, which will allow us to obtain the physical theories, and be the starting point for analysing their physical and geometrical contents.

2.4 Rigid conformal symmetry

Conformal symmetry is defined as the symmetry that preserves angles. Therefore it should contain the transformations that change the metric up to a factor. That implies that the symmetries are determined by the solutions to the 'conformal Killing equation'

$$\partial_{(\mu}\xi_{\nu)}-\tfrac{1}{d}\eta_{\mu\nu}\partial_\rho\xi^\rho=0\,. \tag{2.13}$$

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[3] Note that the scalar here has negative kinetic energy, and the final gravity action has positive kinetic energy.

[4] A gauge fixing can be interpreted as choosing better coordinates such that only one field still transforms under the corresponding transformations. Then, the invariance is expressed as the absence of this field from the action. In this case we would use $g'_{\mu\nu}=g_{\mu\nu}\phi^2$ as D-invariant metric. One can check that this redefinition also leads to (2.12) in terms of the new field.

Exercise 2.2: Get more insight in the meaning of the statement that these are the transformations that preserve 'angles'. To consider angles, we should consider two variations of the same spacetime point. Consider the vectors from x to $x^\mu+(\Delta_1)^\mu$ and another one to $x^\mu+(\Delta_2)^\mu$, where the deformations are considered to be small. The angle between these two is

$$\cos^2\theta = \frac{(\Delta_1\cdot\Delta_2)^2}{(\Delta_1\cdot\Delta_1)\,(\Delta_2\cdot\Delta_2)} \tag{2.14}$$

Now we perform a spacetime transformation that takes a point x to $x'(x) = x+\xi(x)$. Then the first vector will be between $x'(x)$ and $x'(x+\Delta_1) = x'(x) + \Delta_1\cdot\partial x'(x)$. The new vector is thus

$$\Delta_1' = \Delta_1\cdot\partial x'(x) = \Delta_1 + \Delta_1\cdot\partial\xi(x). \tag{2.15}$$

We thus find that

$$\Delta_1'\cdot\Delta_2' = \Delta_1\cdot\Delta_2 + \Delta_1^\rho\left(\partial_\rho\xi^\mu(x)\right)\eta_{\mu\nu}\Delta_2^\nu + \Delta_1^\mu\eta_{\mu\nu}\Delta_2^\rho\left(\partial_\rho\xi^\mu(x)\right), \tag{2.16}$$

where I added indices for clarity. If the last factor gives just a scaling, i.e. if

$$\Delta_1^\rho\left(\partial_\rho\xi^\mu(x)\right)\eta_{\mu\nu}\Delta_2^\nu + \Delta_1^\mu\eta_{\mu\nu}\Delta_2^\rho\left(\partial_\rho\xi^\mu(x)\right) = 2\Lambda_D(x)\Delta_1^\mu\eta_{\mu\nu}\Delta_2^\nu, \tag{2.17}$$

then its easy to see that (2.14) is invariant. Indeed, all factors scale with the same coefficient as the scale factor has to be evaluated at the same spacetime point.

The requirement (2.17) amounts to the scaling of the metric. Indeed, if a metric scales under spacetime transformations,

$$\delta(\mathrm{d}x^\mu\eta_{\mu\nu}\mathrm{d}x^\nu) \equiv 2\mathrm{d}x^\rho\left(\partial_\rho\xi^\mu(x)\right)\eta_{\mu\nu}\mathrm{d}x^\nu = 2\Lambda_D(x)(\mathrm{d}x^\mu\eta_{\mu\nu}\mathrm{d}x^\nu), \tag{2.18}$$

then replace $\mathrm{d}x$ in the above by $(\Delta_1+\Delta_2)$ and subtract the diagonal terms. This leads to (2.17) and hence to the invariance of the angle.

Angles are thus preserved by the transformations that scale the metric and these are the conformal transformations.

In $d=2$ with as non-zero metric elements $\eta_{z\bar z}=1$, the Killing equations are reduced to $\partial_z\xi_z = \partial_{\bar z}\xi_{\bar z} = 0$ and this leads to an infinite dimensional conformal algebra (all holomorphic vectors $\xi_{\bar z}(z)$ and anti-holomorphic vectors $\xi_z(\bar z)$). In dimensions $d>2$ the conformal algebra is finite-dimensional. Indeed, the solutions are

$$\xi^\mu(x) = a^\mu + \lambda_M^{\mu\nu}x_\nu + \lambda_D x^\mu + \left(x^2\Lambda_K^\mu - 2x^\mu x\cdot\Lambda_K\right). \tag{2.19}$$

Corresponding to the parameters a^μ are the translations P_μ, to $\lambda_M^{\mu\nu}$ correspond the Lorentz rotations $M_{\mu\nu}$, to λ_D are associated dilatations D, and Λ_K^μ are parameters of 'special conformal transformations' K_μ. This is expressed as follows for the full set of conformal transformations δ_C:

$$\delta_C = a^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu \,. \tag{2.20}$$

With these transformations, one can obtain the algebra with as non-zero commutators

$$\begin{aligned}
&[M_{\mu\nu}, M^{\rho\sigma}] = -2\delta_{[\mu}^{[\rho} M_{\nu]}{}^{\sigma]} \,, \\
&[P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \,, \qquad [K_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} K_{\rho]} \,, \\
&[P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + 2M_{\mu\nu}) \,, \\
&[D, P_\mu] = P_\mu \,, \qquad\qquad [D, K_\mu] = -K_\mu \,.
\end{aligned} \tag{2.21}$$

This is the SO$(d,2)$ algebra[5]. Indeed one can define

$$M^{\hat\mu\hat\nu} = \begin{pmatrix} M^{\mu\nu} & \frac14(P^\mu - K^\mu) & \frac14(P^\mu + K^\mu) \\ -\frac14(P^\nu - K^\nu) & 0 & -\frac12 D \\ -\frac14(P^\nu + K^\nu) & \frac12 D & 0 \end{pmatrix} \,, \tag{2.22}$$

where indices are raised w.r.t. the rotation matrices $M^{\hat\mu}{}_{\hat\nu}$ with the metric

$$\hat\eta = \text{diag}\,(-1, 1, ..., 1, -1) \,. \tag{2.23}$$

In general, fields $\phi^i(x)$ in d dimensions have the following transformations under the conformal group:

$$\begin{aligned}
\delta_C \phi^i(x) &= \xi^\mu(x)\partial_\mu \phi^i(x) + \Lambda_M^{\mu\nu}(x)\, m_{\mu\nu}{}^i{}_j \phi^j(x) \\
&\quad + \Lambda_D(x)\, k_D^i(\phi) + \Lambda_K^\mu\, (k_\mu \phi)^i\,(x) \,,
\end{aligned} \tag{2.24}$$

where the x-dependent rotation $\Lambda_{M\,\mu\nu}(x)$ and x-dependent dilatation $\Lambda_D(x)$ are given by

$$\begin{aligned}
\Lambda_{M\,\mu\nu}(x) &= \partial_{[\nu}\xi_{\mu]} = \lambda_{M\,\mu\nu} - 4x_{[\mu}\Lambda_{K\,\nu]} \,, \\
\Lambda_D(x) &= \tfrac1d \partial_\rho \xi^\rho = \lambda_D - 2x \cdot \Lambda_K \,.
\end{aligned} \tag{2.25}$$

To specify for each field ϕ^i its transformations under conformal group one has to specify:

i) transformations under the Lorentz group, encoded into the matrix $(m_{\mu\nu})^i{}_j$. The Lorentz transformation matrix $m_{\mu\nu}$ should satisfy

$$m_{\mu\nu}{}^i{}_k m_{\rho\sigma}{}^k{}_j - m_{\rho\sigma}{}^i{}_k m_{\mu\nu}{}^k{}_j = -\eta_{\mu[\rho} m_{\sigma]\nu}{}^i{}_j + \eta_{\nu[\rho} m_{\sigma]\mu}{}^i{}_j \,. \tag{2.26}$$

The explicit form for Lorentz transformation matrices is for vectors (the indices i and j are of the same kind as μ and ν)

$$m_{\mu\nu}{}^\rho{}_\sigma = -\delta_{[\mu}^\rho \eta_{\nu]\sigma} \,, \tag{2.27}$$

while for spinors, (where i and j are (unwritten) spinor indices)

$$m_{\mu\nu} = -\tfrac14 \gamma_{\mu\nu} \,. \tag{2.28}$$

[5]In the 2-dimensional case SO$(2,2)$ = SU$(1,1) \times$ SU$(1,1)$ is realized by the finite subgroup of the infinite dimensional conformal group, and is well known in terms of $L_{-1} = \frac12(P_0 - P_1)$, $L_0 = \frac12 D + M_{10}$, $L_1 = \frac12(K_0 + K_1)$, $\bar L_{-1} = \frac12(P_0 + P_1)$, $\bar L_0 = \frac12 D - M_{10}$, $\bar L_1 = \frac12(K_0 - K_1)$. Higher order $L_n, |n| \geq 2$ have no analogs in $d > 2$.

ii) The dilatational transformation specified by $k_D^i(\phi)$. In most cases (and for all non-scalar fields), we just have

$$k_D^i = w\phi^i\,, \tag{2.29}$$

where w is a real number called the Weyl weight (which is different for each field). However, for scalars in a non-trivial manifold with affine connection $\Gamma_{ij}{}^k$ (torsionless, i.e. symmetric in (ij)), these are solutions of

$$D_i k_D^j \equiv \partial_i k_D^j + \Gamma_{ik}{}^j k_D^k = w\delta_i^j\,, \tag{2.30}$$

where again w is the 'Weyl weight'. For $\Gamma_{ij}{}^k = 0$, this reduces to the simple case (2.29).

iii) Special conformal transformations. Possible extra parts apart from those in (2.19) and (2.25), connected to translations, rotations and dilatations. These are denoted as $(k_\mu\phi)^i$. The commutator between the special conformal transformations and dilations gives the restriction

$$(k_\mu\phi^j)\partial_j k_D^i - k_D^j\partial_j(k_\mu\phi^i) = k_\mu\phi^i\,. \tag{2.31}$$

For the simple form of the dilatations (2.29), this means that $k_\mu\phi^i$ should have Weyl weight one less than that of ϕ^i. Also, for consistency of the $[K,K]$ commutators, the k_μ should be mutually commuting operators.

In this way, the algebra (2.21) is realized on the fields as

$$[\delta_C(\xi_1),\delta_C(\xi_2)] = \delta_C\left(\xi^\mu = \xi_2^\nu\partial_\nu\xi_1^\mu - \xi_1^\nu\partial_\nu\xi_2^\mu\right)\,. \tag{2.32}$$

Note the sign difference between the commutator of matrices (2.26) and the commutator of the generators in (2.21), which is due to the difference between 'active and passive' transformations. To understand fully the meaning of the order of the transformations, consider in detail the calculation of the commutator of transformations of fields. See e.g. for a field of zero Weyl weight ($k_D = 0$), and notice how the transformations act only on fields, not on explicit spacetime points x^μ:

$$\begin{aligned}
\lambda_D a^\mu[D,P_\mu]\phi(x) &= \left(\delta_D(\lambda_D)\delta_P(a^\mu) - \delta_P(a^\mu)\delta_D(\lambda_D)\right)\phi(x)\\
&= \delta_D(\lambda_D)a^\mu\partial_\mu\phi(x) - \delta_P(a^\mu)\lambda_D x^\mu\partial_\mu\phi(x)\\
&= a^\mu\partial_\mu\left(\lambda_D x^\nu\right)\partial_\nu\phi(x)\\
&= a^\mu\lambda_D\partial_\mu\phi(x) = \lambda_D a^\mu P_\mu\phi(x)\,.
\end{aligned} \tag{2.33}$$

It is important to notice that the derivative of a field of Weyl weight w has weight $w+1$. E.g. for a scalar with dilatational transformation determined by the vector k_D (and without extra special conformal transformations) we obtain

$$\begin{aligned}
\delta_C\partial_\mu\phi(x) &= \xi^\nu(x)\partial_\nu\partial_\mu\phi(x) + \Lambda_D(x)\,\partial_\mu k_D\\
&\quad -\Lambda_{M\mu}{}^\nu(x)\partial_\nu\phi(x) + \Lambda_D(x)\partial_\mu\phi(x) - 2\,\Lambda_{K\mu}k_D\,.
\end{aligned} \tag{2.34}$$

The first term on the second line says that it behaves as a vector under Lorentz transformations. Furthermore, this equation implies that the dilatational transformation of $\partial_\mu\phi$

is determined by $\partial_\mu(k_D+\phi)$. For the simple transformation (2.29), this means that the derivative also satisfies such a simple transformation with weight $w+1$. Furthermore, the derivative $\partial_\mu\phi$ has an extra part in the special conformal transformation of the form $\Lambda_K^\nu k_\nu(\partial_\mu\phi)=-2\Lambda_{K\mu}k_D$.

With these rules the conformal algebra is satisfied. The question remains when an action is conformal invariant. We consider local actions which can be written as $S=\int d^dx\,\mathcal{L}(\phi^i(x),\partial_\mu\phi^i(x))$, i.e. with at most first order derivatives on all the fields. For P_μ and $M_{\mu\nu}$ there are the usual requirements of a covariant action. For the dilatations, let us for now just take for all fields the simple law (2.29) with w_i as weight for each field ϕ^i (for an extension, see exercise 2.4). Then we have the requirement that the weights of all fields in each term should add up to d, where ∂_μ counts also for 1, as can be seen from (2.34). Indeed, the explicit Λ_D transformations finally have to cancel with

$$\xi^\mu(x)\partial_\mu\mathcal{L}\approx-(\partial_\mu\xi^\mu(x))\mathcal{L}=-d\Lambda_D(x)\mathcal{L}\,. \tag{2.35}$$

For special conformal transformations one remains with

$$\delta_K S=2\Lambda_K^\mu\int \mathrm{d}^dx\,\frac{\mathcal{L}\overleftarrow{\partial}}{\partial(\partial_\nu\phi^i)}\left(-\eta_{\mu\nu}w_i\phi^i+2m_{\mu\nu}{}^i{}_j\phi^j\right)+\Lambda_K^\mu\frac{S\overleftarrow{\delta}}{\delta\phi^i(x)}(k_\mu\phi)^i(x)\,. \tag{2.36}$$

where $\overleftarrow{\partial}$ indicates a right derivative. The first terms originate from the K-transformations contained in (2.19) and (2.25). In most cases these are sufficient to find the invariance and no $(k_\mu\phi)$ are necessary. In fact, the latter are often excluded because of the requirement that they should have Weyl weight w_i-1, and in many cases there are no such fields available.

Exercise 2.3: There are typical cases in which (2.36) does not receive any contributions. Check the following ones

1. scalars with Weyl weight 0.
2. spinors appearing as $\partial\!\!\!/\lambda$ if their Weyl weight is $(d-1)/2$. This is also the appropriate weight for actions as $\bar\lambda\partial\!\!\!/\lambda$.
3. Vectors or antisymmetric tensors whose derivatives appear only as field strengths $\partial_{[\mu_1}B_{\mu_2\ldots\mu_p]}$ if their Weyl weight is $p-1$. This value of the Weyl weight is what we need also in order that their gauge invariances and their zero modes commute with the dilatations. Then scale invariance of the usual square of the field strengths will fix $p=\frac{d}{2}$.
4. Scalars X^i with Weyl weight $\frac{d}{2}-1$ and

$$\mathcal{L}=(\partial_\mu X^i)A_{ij}(\partial^\mu X^j)\,, \tag{2.37}$$

where A_{ij} are constants.

Exercise 2.4: When the scalars transform under dilatations and special conformal transformations according to

$$\delta\phi^i = \xi^\mu(x)\partial_\mu\phi^i + \Lambda_D(x)k^i_D(\phi)\,, \tag{2.38}$$

with $k^i_D(\phi)$ arbitrary, check that the conformal algebra is satisfied. Consider now the action for scalars

$$S_{\rm sc} = -\tfrac{1}{2}\int {\rm d}^d x\, \partial_\mu\phi^i\, g_{ij}(\phi)\, \partial^\mu\phi^j\,. \tag{2.39}$$

It is invariant under translations and Lorentz rotations. Check that the dilatational and special conformal transformations leave us with

$$\begin{aligned}\delta S_{\rm sc} \;=& -\textstyle\int {\rm d}^d x \quad \left\{\Lambda_D(x)\partial_\mu\phi^i\,\partial^\mu\phi^j\left[g_{k(i}\left(\partial_{j)}k^k_D + \Gamma_{j\ell}{}^k k^\ell_D\right) - \tfrac{1}{2}(d-2)g_{ij}\right]\right.\\ &\left. -2\Lambda^\mu_K\partial_\mu\phi^i\, g_{ij}k^j_D\right\}\,,\end{aligned} \tag{2.40}$$

if one identifies the affine connection with the Levi-Civita connection of the metric [similar to (A.4)]. The invariance under rigid dilatations is thus already obtained if

$$D_{(i}k_{j)D} = \tfrac{1}{2}(d-2)g_{ij}\,, \tag{2.41}$$

with the usual definition of a covariant derivative D_i. Vectors satisfying this equation are called 'homothetic Killing vectors'[6]. However, to obtain special conformal invariance, the last term of (2.40), originating from a contribution $\partial_\mu\Lambda_D(x)$, should be a total derivative. Thus one finds that

$$k_{iD} = \partial_i k\,, \tag{2.42}$$

for some k. Then $D_i k_{jD}$ is already symmetric by itself, and thus (2.41) implies (2.30) with $w = (d-2)/2$. Thus the vectors satisfying (2.30) are 'exact homothetic Killing vectors', and moreover the exactness for a homothetic Killing vector is sufficient to prove (2.30). One can find that

$$(d-2)k = k^i_D g_{ij} k^j_D\,. \tag{2.43}$$

A systematic investigation of conformal actions for scalars in gravity can be found in [22].

Exercise 2.5: Check that for a Lagrangian of the form

$$\mathcal{L} = (\partial_\mu\phi^1)(\partial^\mu\phi^2)\,, \tag{2.44}$$

the conformal Killing equation has as solution

$$k^1_D = w_1\phi^1\,, \qquad k^2_D = w_2\phi^2\,, \qquad w_1 + w_2 = d-2\,. \tag{2.45}$$

[6] The terminology reflects that the right hand side is a *constant* times g_{ij}. For an function times g_{ij} it is just a 'conformal Killing vector'.

However, the equation (2.30) gives also that $w_1 = w_2 = \frac{1}{2}(d-2)$. Check that this is necessary for special conformal transformations.

Modifying the Lagrangian to

$$\mathcal{L} = \left(1 + \frac{\phi^1}{\phi^2}\right)(\partial_\mu \phi^1)(\partial^\mu \phi^2)\,, \tag{2.46}$$

one finds as only non-zero Levi–Civita connections

$$\Gamma^1_{11} = \frac{1}{\phi^1 + \phi^2}\,, \qquad \Gamma^2_{22} = -\frac{\phi^1/\phi^2}{\phi^1 + \phi^2}\,. \tag{2.47}$$

The solutions of the conformal Killing equations (2.41) already fix (2.45) with $w_1 = w_2 = \frac{1}{2}(d-2)$. However, now the equation (2.30) gives no solution. Hence, this model can only have rigid dilatations, but no rigid special conformal transformations.

2.5 Superconformal groups

A classical work on the classification of superconformal groups is the paper of W. Nahm [23]. In the groups that he classified, the bosonic subgroup is a direct product of the conformal group and the R-symmetry group. It implies that bosonic symmetries that are not in the conformal algebra are spacetime scalars. This was motivated by the works of [21] and [24], who had obtained such a condition from non-triviality of scattering amplitudes. However, with branes the assumptions of these papers may too constrained. Other examples have been considered first in 10 and 11 dimensions in [25]. Recently, a new classification has appeared in [26] from which we can extract[7] table 2[8] for dimensions from 3 to 11. The bosonic subgroup contains always two factors. One contains the conformal group. If that factor is really the conformal group, then the algebra appears in Nahm's classification. Note that 5 dimensions is a special case. There is a generic superconformal algebra for any extension. But for the case $N = 2$ there exists a smaller superconformal algebra that is in Nahm's list. Note that for $D = 6$ or $D = 10$, where one can have chiral spinors, only the case that all supersymmetries have the same chirality has been included. Non-chiral supersymmetry can be obtained from the reduction in one more dimension, so e.g. from the $d = 7$ algebra $\mathrm{OSp}(16^*|N)$ we obtain (N, N) in $d = 6$ [27]. So far, superconformal tensor calculus has only been based on algebras of Nahm's type.

We thus see that for $N = 2$ in $d = 4$, $d = 5$ and $d = 6$ the superconformal algebras are respectively $\mathrm{SU}(2,2|2)$, $\mathrm{F}^2(4)$ and $\mathrm{OSp}(8^*|2)$. But, in the practical treatment we will not see any fundamental difference in the structure of the supergroups. The bosonic subgroups are respectively $\mathrm{SO}(4,2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, $\mathrm{SO}(5,2)\times\mathrm{SU}(2)$ and $\mathrm{SO}(6,2)\times\mathrm{SU}(2)$. The first factor (one should consider its covering that allows the spinor representation) is the conformal group, which consists of translations P_a, Lorentz rotations M_{ab}, dilatations D, and special

[7] For the notations on groups and supergroups, see appendix B.
[8] See appendix B for the notations of groups and supergroups.

Table 2: *Superconformal algebras, with the two parts of the bosonic subalgebra: one that contains the conformal algebra and the other one is the R-symmetry. In the cases* $d = 4$ *and* $d = 8$*, the* $U(1)$ *factor in the R-symmetry group can be omitted for* $N \neq 4$ *and* $N \neq 16$*, respectively.*

d	supergroup	bosonic group	
3	$\mathrm{OSp}(N\vert 4)$	$\mathrm{Sp}(4) = \mathrm{SO}(3,2)$	$\mathrm{SO}(N)$
4	$\mathrm{SU}(2,2\vert N)$	$\mathrm{SU}(2,2) = \mathrm{SO}(4,2)$	$\mathrm{SU}(N)\times \mathrm{U}(1)$
5	$\mathrm{OSp}(8^*\vert N)$	$\mathrm{SO}^*(8) \supset \mathrm{SO}(5,2)$	$\mathrm{USp}(N)$
	$\mathrm{F}(4)$	$\mathrm{SO}(5,2)$	$\mathrm{SU}(2)$
6	$\mathrm{OSp}(8^*\vert N)$	$\mathrm{SO}^*(8) = \mathrm{SO}(6,2)$	$\mathrm{USp}(N)$
7	$\mathrm{OSp}(16^*\vert N)$	$\mathrm{SO}^*(16) \supset \mathrm{SO}(7,2)$	$\mathrm{USp}(N)$
8	$\mathrm{SU}(8,8\vert N)$	$\mathrm{SU}(8,8) \supset \mathrm{SO}(8,2)$	$\mathrm{SU}(N)\times \mathrm{U}(1)$
9	$\mathrm{OSp}(N\vert 32)$	$\mathrm{Sp}(32) \supset \mathrm{SO}(9,2)$	$\mathrm{SO}(N)$
10	$\mathrm{OSp}(N\vert 32)$	$\mathrm{Sp}(32) \supset \mathrm{SO}(10,2)$	$\mathrm{SO}(N)$
11	$\mathrm{OSp}(N\vert 64)$	$\mathrm{Sp}(64) \supset \mathrm{SO}(11,2)$	$\mathrm{SO}(N)$

conformal transformations K_a. The remaining part of the bosonic group is the so-called R-symmetry group, the automorphism group of the supersymmetries. The fermionic generators are the supersymmetries Q^i and the special supersymmetries S^i. In 6 dimensions the S^i have opposite chirality than the Q^i, thus

$$S^i = \gamma_7 S^i\,. \tag{2.48}$$

In 4 dimensions, I also use the opposite convention for the chirality, i.e. (compare with (2.1))

$$S^i = \gamma_5 S^i\,, \qquad S_i = -\gamma_5 S_i\,. \tag{2.49}$$

The dilatations provide a 3-grading of the conformal algebra and a 5-grading of the superconformal algebra. In the conformal algebra, the translations have weight 1, in the sense that $[D, P_a] = P_a$. The special conformal transformations have weight -1, i.e. $[D, K_a] = -K_a$, and the Lorentz generators and dilatations commute, i.e. they have all weight 0. Also the R-symmetry algebra has weight 0. The supersymmetries have weight $+\frac{1}{2}$ (see already that this is consistent with (2.4)), and the special supersymmetries have weight $-\frac{1}{2}$. Thus, there is a clear structure in the superconformal algebra, ordering them according to the Weyl (dilatational) weight:

$$\begin{array}{rcl} 1 & : & P_\mu \\ \frac{1}{2} & : & Q \\ 0 & : & D,\ M_{ab},\ SU(2),\ (U(1)) \\ -\frac{1}{2} & : & S \\ -1 & : & K_\mu\,, \end{array} \tag{2.50}$$

Many of the commutators are determined already by what was said. We have for all the dimensions as commutation relations[9]

$$\begin{array}{rrcl}
 & [M_{ab},M_{cd}] &=& \eta_{a[c}M_{d]b}-\eta_{b[c}M_{d]a}\,,\\
 & [P_a,M_{bc}] &=& \eta_{a[b}P_{c]}\,,\qquad [K_a,M_{bc}] \;=\; \eta_{a[b}K_{c]}\,,\\
 & [D,P_a] &=& P_a\,,\qquad [D,K_a] \;=\; -K_a\,,\\
 & [P_a,K_b] &=& 2(\eta_{ab}D+2M_{ab})\,,\\
 & [U_i{}^j,U_k{}^\ell] &=& \delta_i{}^\ell U_k{}^j-\delta_k{}^jU_i{}^\ell\,,\\
 & [M_{ab},Q^i_\alpha] &=& -\tfrac14(\gamma_{ab}Q^i)_\alpha\,,\qquad [M_{ab},S^i_\alpha] \;=\; -\tfrac14(\gamma_{ab}S^i)_\alpha\,,\\
 & [D,Q_\alpha{}^i] &=& \tfrac12Q_\alpha{}^i\,,\qquad [D,S_\alpha{}^i]=-\tfrac12S^i_\alpha\,,\\
 & [U_i{}^j,Q_\alpha{}^k] &=& \delta_i{}^kQ^j_\alpha-\tfrac12\delta_i{}^jQ_\alpha{}^k\,,\qquad [U_i{}^j,S_\alpha{}^k] \;=\; \delta_i{}^kS^j_\alpha-\tfrac12\delta_i{}^jS_\alpha{}^k\,,\\
d=4\ : & [U(1),Q_\alpha{}^i] &=& -\mathrm{i}\tfrac12Q_\alpha{}^i\,,\qquad [U(1),S_\alpha{}^i] \;=\; -\mathrm{i}\tfrac12S_\alpha{}^i\\
d=4\ : & [K_a,Q^i_\alpha] &=& (\gamma_aS^i)_\alpha\,,\qquad [P_a,S_\alpha{}^i] \;=\; (\gamma_aQ^i)_\alpha\,,\\
d=5\ : & [K_a,Q^i_\alpha] &=& \mathrm{i}(\gamma_aS^i)_\alpha\,,\qquad [P_a,S_\alpha{}^i] \;=\; -\mathrm{i}(\gamma_aQ^i)_\alpha\,,\\
d=6\ : & [K_a,Q^i_\alpha] &=& -(\gamma_aS^i)_\alpha\,,\qquad [P_a,S_\alpha{}^i] \;=\; -(\gamma_aQ^i)_\alpha\,.
\end{array}\tag{2.51}$$

The factors i are necessary in 5 dimensions if we use the same symplectic Majorana condition for all the spinors. This can be seen easily from the C-conjugation rules in appendix A.2.2.

The R-symmetry group is a compact group. In the standard conventions, avoiding artificial i factors in group theory, generators corresponding to a compact action are anti-Hermitian. For the SU(2) generators we thus have

$$\begin{array}{rcl}
U_i{}^j &=& -\left(U_j{}^i\right)^*\,,\qquad U_i{}^i=0\,,\\
U_i{}^j &=& \mathrm{i}(U_1\sigma_1+U_2\sigma_2+U_3\sigma_3)\,,
\end{array}\tag{2.52}$$

where U_1, U_2 and U_3 are real operators. The equations of the first line are valid as well, on the one hand in $d=4$, where raising and lowering of i-indices is done by complex conjugation, and on the other hand in $d=5,6$ where they are raised and lowered by the ε^{ij} as in (A.39). In 4 dimensions, the first equation is written as

$$d=4\ :\ U_i{}^j=-U^j{}_i\,.\tag{2.53}$$

In 5 and 6 dimensions the one-index raised or lowered matrices are (e.g. $\sigma_{1ij}=\sigma_{1i}{}^k\varepsilon_{kj}=\sigma_1\mathrm{i}\sigma_2=-\sigma_3$)

$$d=5,6\ :\ U_{ij}=-\mathrm{i}U_1\sigma_3-U_2\mathbb{1}+\mathrm{i}\sigma_1U_3\,,\qquad U^{ij}=\mathrm{i}U_1\sigma_3-U_2\mathbb{1}-\mathrm{i}\sigma_1U_3\,.\tag{2.54}$$

The first equation of (2.52) implies with the C-rule $M^C=\sigma_2M^*\sigma_2$ that U is a C-invariant matrix. The tracelessness translates in symmetry of $U_{(ij)}$ and $U^{(ij)}$. Thus, in conclusion, U is C-invariant and symmetric (and thus also $U_i{}^j=U^j{}_i$ here, different from (2.53)).

[9] To prepare these formulae, I made use of [28] for $d=4$, apart from a sign change in the choice of charge conjugation, such that the anticommutators of fermionic generators have all opposite sign. For $d=5$, I made use of [29], and for $d=6$ of [30], but replacing there K by $-K$ and U_{ij} by $-\frac12U_{ij}$.

Exercise 2.6: Check that in 5 and 6 dimensions the $[U,Q]$ commutator can be written as $[U_{ij},Q^k]=\delta_{[i}{}^kQ_{j)}$.

In 4 dimensions, the R-symmetry group is $\mathrm{SU}(2)\times\mathrm{U}(1)$. I write U(1) as a real generator.

The anticommutation relations between the fermionic generators are

$$\begin{array}{rll}
d=4 & : & \{Q_\alpha{}^i,Q_j{}^\beta\} = -\tfrac12\delta_j{}^i(\gamma^a)_\alpha{}^\beta P_a\,, \qquad \{Q_\alpha{}^i,Q^{j\beta}\} = 0\,,\\
 & & \{S_\alpha{}^i,S_j{}^\beta\} = -\tfrac12\delta_j{}^i(\gamma^a)_\alpha{}^\beta K_a\,, \qquad \{S_\alpha{}^i,S^{j\beta}\} = 0\,, \qquad \{Q_\alpha{}^i,S^{j\beta}\} = 0\,,\\
 & & \{Q_\alpha{}^i,S_j{}^\beta\} = -\tfrac12\delta_j{}^i\delta_\alpha{}^\beta D-\tfrac12\delta_j{}^i(\gamma^{ab})_\alpha{}^\beta M_{ab}+\mathrm{i}\tfrac12\delta_j{}^i\delta_\alpha{}^\beta U(1)+\delta_\alpha{}^\beta U_j{}^i\,,\\
d=5,6 & : & \{Q_{i\alpha},Q^{j\beta}\} = -\tfrac12\delta_i{}^j(\gamma^a)_\alpha{}^\beta P_a\,, \qquad \{S_{i\alpha},S^{j\beta}\} = -\tfrac12\delta_i{}^j(\gamma^a)_\alpha{}^\beta K_a\,,\\
d=5 & : & \{Q_{i\alpha},S^{j\beta}\} = -\mathrm{i}\tfrac12\left(\delta_i{}^j\delta_\alpha{}^\beta D+\delta_i{}^j(\gamma^{ab})_\alpha{}^\beta M_{ab}+3\delta_\alpha{}^\beta U_i{}^j\right)\,,\\
d=6 & & \{Q_{i\alpha},S^{j\beta}\} = \tfrac12\left(\delta_i{}^j\delta_\alpha{}^\beta D+\delta_i{}^j(\gamma^{ab})_\alpha{}^\beta M_{ab}+4\delta_\alpha{}^\beta U_i{}^j\right)\,.
\end{array} \tag{2.55}$$

Note that in 6 dimensions, the spinors have a chiral projection, and the gamma matrices in the right hand side should also be understood as their chiral projection, thus e.g. in the last line $\delta_\alpha{}^\beta$ stands for $\tfrac12(\mathbb{1}-\gamma_7)_\alpha{}^\beta$.

2.6 Rules of (super)gauge theories, gauge fields and curvatures.

Consider a general (super)algebra with commutators

$$[\delta_A(\epsilon_1^A),\delta_B(\epsilon_2^B)]=\delta_C\left(\epsilon_2^B\epsilon_1^Af_{AB}{}^C\right)\,. \tag{2.56}$$

In general the $f_{AB}{}^C$ may be structure functions, i.e. depend on the fields. The equality may be true only modulo equations of motion, as we will see below. To start with, the $f_{AB}{}^C$ are related to the abstract algebra introduced in subsection 2.5:

$$\begin{array}{rll}
d=4,6 & : & [T_A,T_B\}=T_AT_B-(-)^{AB}T_BT_A=f_{AB}{}^CT_C\,,\\
d=5 & : & [T_A,T_B\}=T_AT_B-(-)^{AB}T_BT_A=(-)^{AB}f_{AB}{}^CT_C\,,
\end{array} \tag{2.57}$$

where $(-)^A$ is a minus sign if T_A is fermionic. The extra sign factor for $d=5$ is due to the factors i in (2.2). It is assumed that in all other cases the transformation is generated by ϵ^AT_A. Soon, however, the structure constants will be changed in structure functions.

To realize such an algebra one needs of course gauge fields and curvatures. This means that for every generator there is a gauge field $h_\mu{}^A$ with

$$\delta(\epsilon)h_\mu{}^A=\partial_\mu\epsilon^A+\epsilon^Ch_\mu{}^Bf_{BC}{}^A\,. \tag{2.58}$$

Note that the order of the fields and parameters is relevant here. For fermionic fields, the indices contain spinor indices and one may use the conventions of section A.3.1. Although the objects may be fermionic or bosonic, you do not see many sign changes. The trick to avoid most of the signs is to keep objects with contracted indices together. E.g., you see here the B index of the gauge field next to the B index in the structure constants, and then the C contracted indices do not have another uncontracted index.

Covariant derivatives have a term involving the gauge field for every gauge transformation,

$$\nabla_\mu = \partial_\mu - \delta_A(h_\mu{}^A)\,, \tag{2.59}$$

and their commutators are new transformations with as parameters the curvatures:

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] &= -\delta_A(R_{\mu\nu}{}^A), \\ R_{\mu\nu}{}^A &= 2\partial_{[\mu} h_{\nu]}{}^A + h_\nu{}^C h_\mu{}^B f_{BC}{}^A\,, \end{aligned} \tag{2.60}$$

which transform 'covariantly' as

$$\delta R_{\mu\nu}{}^A = \epsilon^C R_{\mu\nu}{}^B f_{BC}{}^A\,. \tag{2.61}$$

They satisfy Bianchi identities, which are, with the definition (2.59),

$$\nabla_{[\mu} R_{\nu\rho]}{}^A = 0\,. \tag{2.62}$$

Exercise 2.7: Check that the first equation of (A.2) corresponds to the above definition for $R_{\mu\nu}{}^{ab}$ if we just consider the Lorentz group, i.e. the first equation of (2.51), and define $\omega_\mu{}^{ab}$ as the gauge field of M_{ab}.

3 Gauging spacetime symmetries and the Weyl multiplet.

The rules of gauge theories given in section 2.6 are easy. I can start by giving a name to the gauge fields and parameters for the generators that appeared in section 2.5:

P_a	M_{ab}	D	K_a	$U_i{}^j$	$U(1)$	Q	S
e_μ^a	$\omega_\mu{}^{ab}$	b_μ	$f_\mu{}^a$	$V_{\mu j}{}^i$	A_μ	ψ_μ	ϕ_μ
ξ^a	λ^{ab}	Λ_D	Λ_K^a	$\Lambda_j{}^i$	Λ_A	ϵ	η

Note that the table is made for $d=4$ and the conventions in $d=4$ and $d=5,6$ are different for the relation between the SU(2) generators and parameters (translating to gauge fields)[10]:

$$d=4\ :\ \delta = \cdots + \Lambda_i{}^j U_j{}^i\,, \qquad d=5,6\ :\ \delta = \cdots + \Lambda^{ij} U_{ij}\,. \tag{3.1}$$

However, these rules do not lead to a suitable theory for spacetime symmetries. There are several shortcomings if we straightforwardly apply the previous procedure:

1. $\omega_\mu{}^{ab}$ should not be an independent field in general relativity. It is a function of e_μ^a.
2. General coordinate transformations should appear rather than local translations.
3. The number of bosonic and fermionic components do not match.

[10] The different normalization of U w.r.t. [30], see footnote 9, implies that $V_\mu{}^{ij}$ in that paper has to be replaced by $-2V_\mu{}^{ij}$ to get the formulae here.

3.1 Gauge theory of spacetime symmetries

As in superspace, we need constraints to get to the theory that we want. These will be given in section 3.3. However, there are more conceptual changes necessary. First of all I will discuss the essential modifications with respect to what was presented in the previous section.

1. Translations will be replaced by general coordinate transformations (gct). These gct will have to be distinguished from the other transformations.

2. The structure constants will be replaced by structure functions.

3. The multiplet that gauges the superconformal group contains also fields that are not gauge fields.

4. Some gauge fields are 'composite', i.e. functions of the other fields in the multiplet.

Because of the distinguished role of the gct, we split the range of indices A in the translations, P_a, and the remaining ones, which are called T_I. The gauge field of the former is at once denoted as $e_\mu{}^a$, i.e. the vielbein. One requires this field to be invertible as a matrix. The translations are replaced by 'covariant general coordinate transformations'. These are by definition [31, 32]

$$\delta_{\rm cgct}(\xi) = \delta_{\rm gct}(\xi) - \delta_I(\xi^\mu h_\mu{}^I)\,, \tag{3.2}$$

a combination of general coordinate transformations and all the non-translation transformations whose parameter ϵ^I is replaced by $\xi^\mu h_\mu{}^I$. As we want the final action to be invariant under general coordinate transformations and under all the transformations labelled by I, the action should be invariant also under covariant general coordinate transformations.

Further, I make a distinction between 'gauge fields' and matter fields. Gauge fields have a local spacetime index μ. Of course one could change it to an index a by multiplication with $e_a{}^\mu$, but I consider it in the form h_μ^A as a basic field. This basic field may still be composite, but that is not important at this point. Matter fields have no local spacetime index. The fundamental difference between gauge fields and matter fields is that the transformation of matter fields does not involve a derivative of a gauge parameter, while the transformation of gauge fields have the $\partial_\mu\epsilon^A$ term.

I now present the general form of the transformations.

3.1.1 Transformations of the vielbeins

Remark the following expression for gct on gauge fields

$$\delta_{\rm gct}(\xi)h_\mu{}^A \equiv \xi^\nu\partial_\nu h_\mu{}^A + (\partial_\mu\xi^\nu)h_\nu{}^A = \delta_B(\xi^\nu h_\nu{}^B)h_\mu{}^A - \xi^\nu R_{\mu\nu}{}^A\,. \tag{3.3}$$

For a covariant general coordinate transformation (cgct) the last term of (3.2) cancels the first term here, apart from the translation part, i.e. where B takes only the values corresponding to translations. On the vielbeins this gives for the covariant general coordinate transformation (cgct)

$$\delta_{\rm cgct}(\xi)e_\mu{}^a = \partial_\mu\xi^a + \xi^b h_\mu{}^B f_{Bb}{}^a - \xi^\nu R_{\mu\nu}(P^a)\,, \tag{3.4}$$

where the first two terms are (2.58) for the parameter ϵ replaced by the translation parameter ξ^a, and used for e_μ^a as gauge field of translations. At this point, I introduce a first constraint:

$$R_{\mu\nu}(P^a) = 2\left(\partial_{[\mu} + b_{[\mu}\right) e_{\nu]}^a + 2\omega_{[\mu}{}^{ab} e_{\nu]b} + \xi^a(\psi_\mu, \psi_\nu) = 0\,, \tag{3.5}$$

where ξ^a is the function introduced in (2.6). This is the constraint that will imply that $\omega_\mu{}^{ab}$ is a Levi–Civita connection such that for pure gravity the spacetime manifold with metric $g_{\mu\nu} = e_\mu^a e_{\nu a}$ is torsionless. I will come back to this constraint and its consequences below, and explain its solution for $\omega_\mu{}^{ab}$ in general. With this constraint, the covariant general coordinate transformation of the vielbein is just its P_a transformation as it would directly follow from (2.58):

$$\delta_{\rm cgct}(\xi) e_\mu{}^a = (\partial_\mu + b_\mu)\xi^a + \omega_\mu{}^{ab}\xi_b\,. \tag{3.6}$$

I used the explicit expressions of the commutators that are of the form $[P, \cdot] = P$.

The other transformations of the vielbein follow straightforwardly from the rule (2.58). I.e., using the notation $\xi^a(\cdot,\cdot)$ from (2.6),

$$\begin{aligned} \delta_I(\epsilon^I) e_\mu^a &= \epsilon^I h_\mu^B f_{BI}{}^a \\ &= -\Lambda_D e_\mu^a - \lambda^{ab} e_{\mu b} + \xi^a(\psi_\mu, \epsilon)\,, \\ d = 4 \quad &: \quad \xi^a(\psi_\mu, \epsilon) = \tfrac12 \bar\epsilon^i \gamma^a \psi_{\mu i} + \tfrac12 \bar\epsilon_i \gamma^a \psi_\mu^i\,, \\ d = 5, 6 \quad &: \quad \xi^a(\psi_\mu, \epsilon) = \tfrac12 \bar\epsilon^i \gamma^a \psi_{\mu i}\,. \end{aligned} \tag{3.7}$$

3.1.2 Transformations of the other gauge fields

For the other gauge fields, I will allow a modification of (2.58), due to the foreseen presence of matter fields in the multiplet. I consider the following general form of I-type gauge transformations:

$$\begin{aligned} \delta_J(\epsilon^J) h_\mu^I &= \partial_\mu \epsilon^I + \epsilon^J h_\mu{}^A f_{AJ}{}^I + \epsilon^J M_{\mu J}{}^I \\ &= \partial_\mu \epsilon^I + \epsilon^J h_\mu{}^K f_{KJ}{}^I + \epsilon^J \mathcal{M}_{\mu J}{}^I, \\ \mathcal{M}_{aJ}{}^I &= M_{aJ}{}^I + f_{aJ}{}^I\,. \end{aligned} \tag{3.8}$$

The expression $M_{aJ}{}^I$ is a function of 'matter fields'. E.g. for super-Maxwell theories, the field h_μ^I can be the U(1) gauge field W_μ. The multiplet then contains also a 'gaugino', Ω^i, which has to appear in the transformation of the gauge vector. This is the term ($d = 4$)

$$\delta_Q(\epsilon) W_\mu = -\tfrac12 \varepsilon_{ij} \bar\epsilon^i \gamma_\mu \Omega^j + \text{h.c.}\,, \tag{3.9}$$

which is thus of the form of an M-term in (3.8). But also in the Weyl multiplet there will be several terms like this, involving the extra matter fields of the multiplet. $M_{aJ}{}^I$ should be a covariant quantity. I will come back to covariant quantities below, but for now it is any quantity that does not transform with a derivative. I will come back to this issue below.

The extra terms determined by ${f_{aJ}}^I$ can be given explicitly:

$$\begin{array}{rcl}\delta_I(\epsilon^I)\psi_\mu^i & = & \ldots + s_d\gamma_\mu\eta^i \qquad \text{with} \qquad \left\{\begin{array}{lcl} d=4 & : & s_4=-1,\\ d=5 & : & s_5=-\mathrm{i},\\ d=6 & : & s_6=1.\end{array}\right.\\ \delta_I(\epsilon^I)b_\mu & = & \ldots + 2\Lambda_{K\mu},\\ \delta_I(\epsilon^I){\omega_\mu}^{ab} & = & \ldots - 4\Lambda_K^{[a}{e_\mu}^{b]}\,.\end{array} \tag{3.10}$$

To determine the sign factors from the algebra, you have to use symmetry properties of the gamma matrix and charge conjugation.

The covariant general coordinate transformations on the gauge fields are from the definition (3.2) and the transformations (3.8)

$$\begin{array}{rcl}\delta_{\rm cgct}(\xi)h_\mu^I & = & -\xi^\nu {R_{\mu\nu}}^I + \xi^a h_\mu^J {f_{Ja}}^I - \xi^\nu h_\nu^J {M_{\mu J}}^I\\ & = & -\xi^\nu {\widehat{R}_{\mu\nu}}{}^I - \xi^a h_\mu^J {\mathcal{M}_{aJ}}^I\,.\end{array} \tag{3.11}$$

The second term in the first line only occurs for the transformation of the gauge fields of supersymmetry, dilatations and Lorentz rotations. This is the original P_a transformation of the gauge field. In the second line appears a new covariant curvature, which takes the transformations of the gauge fields to matter fields into account. Indeed, the last term of the first line of (3.8) implies that ${R_{\mu\nu}}^I$ transforms in the derivative of a parameter, i.e. there is a term $2\partial_{[\mu}\epsilon^J {M_{\nu]J}}^I$. The modified curvature

$${\widehat{R}_{\mu\nu}}{}^I = {R_{\mu\nu}}^I - 2{h_{[\mu}}^J {M_{\nu]J}}^I = {r_{\mu\nu}}^I - 2{h_{[\mu}}^J {\mathcal{M}_{\nu]J}}^I\,, \tag{3.12}$$

does not transform to a derivative of a parameter. In the last expression, I have extracted all contributions from translations out of the curvatures. So

$${r_{\mu\nu}}^I = 2\partial_{[\mu}{h_{\nu]}}^I + {h_\nu}^K {h_\mu}^J {f_{JK}}^I\,. \tag{3.13}$$

Exercise 3.1: We saw already the explicit form of the terms that make the difference between M and $\mathcal{M}$. This should allow you to determine that the only ones where these play a role are

$$\begin{array}{rcl}R_{\mu\nu}(M^{ab}) & = & r_{\mu\nu}(M^{ab}) + 8{f_{[\mu}}^{[a}{e_{\nu]}}^{b]},\\ R_{\mu\nu}(D) & = & r_{\mu\nu}(D) - 4{f_{[\mu}}^a e_{\nu]a},\\ R_{\mu\nu}(Q) & = & r_{\mu\nu}(Q) + 2s_d\gamma_{[\mu}\phi_{\nu]}\,.\end{array} \tag{3.14}$$

I have not discussed here how the translations are deformed to these covariant general coordinate transformations. This, I could do for the vielbein, imposing the constraint (3.5). However, for the other gauge field this is not a straightforward step. First of all, we cannot simply put all curvatures equal to zero to use (3.3). This would impose derivative constraints on all fields, and thus restrict its dynamics. It could be done for $R(P)$ because in that case, this constraint will just determine the spin connection in terms of the vielbein, without imposing further dynamical constraints. The solution differs for the different gauge fields. That is the reason why we need the matter fields in the transformations (3.8). An illustration how these covariant general coordinate transformations do appear in the algebra, rather than the translations, will be given in section 3.2.2. The modifications that are necessary to get the covariant general coordinate transformations for the gauge fields of the Weyl multiplet will be the subject of section 3.3.2.

3.1.3 Transformations of matter fields

Finally there are the matter fields. Their I-type transformations should be 'covariant expressions', to which we turn in a moment. The covariant general coordinate transformations, defined by (3.2), give

$$\delta_{\rm cgct}(\xi)\phi = \xi^\mu\partial_\mu\phi - \delta_I(\xi^\mu h_\mu^I)\phi\,. \tag{3.15}$$

As the parameter of transformations for matter fields by definition appears without a derivative, I can write

$$\delta_{\rm cgct}(\xi)\phi = \xi^\mu D_\mu\phi\,,\qquad D_\mu\phi \equiv \partial_\mu\phi - \delta_I(h_\mu^I)\phi\,. \tag{3.16}$$

This defines the new covariant derivative. The definition of ∇ in (2.59) leads to a vanishing result when identifying $P_\mu = e_\mu^a P_a$ with the generator of covariant general coordinate transformations:

$$\nabla_\mu\phi = \partial_\mu\phi - e_\mu^a P_a\phi - \delta_I(h_\mu^I)\phi = 0\,. \tag{3.17}$$

Thus on matter fields we have $P_a\phi = D_a\phi$.

3.2 Covariant quantities and covariant derivatives

I already used the word 'covariant quantity'.
Definition of a covariant quantity. *A covariant quantity is a field whose transformation under any local symmetry has no derivative on a transformation parameter.*
In particular, as its general coordinate transformation should not involve a derivative on the parameter ξ^μ, it has to be a world scalar.

There are two ways to build new covariant quantities:

1. A covariant derivative on a covariant quantity with its index turned to a tangent spacetime index:

$$D_a\phi = e_a{}^\mu D_\mu\phi\,. \tag{3.18}$$

2. Covariant curvatures with their indices turned to tangent spacetime indices:

$$\widehat{R}_{ab}{}^I = e_a{}^\mu e_b{}^\nu \widehat{R}_{\mu\nu}{}^I\,. \tag{3.19}$$

The elementary matter fields transform under the symmetries in other covariant quantities. Let me first warn you that this is not a general property of covariant quantities, as we will see below. For quantities that do not transform in covariant quantities, the expression of the covariant derivative is not as in (3.16). Let us first look to covariant quantities that do transform in covariant quantities. For those we have the theorem

Theorem on covariant derivatives. *If a covariant quantity ϕ transforms only into covariant quantities under I-transformations, then its covariant derivative (3.18) with $D_\mu\phi$ given by (3.16) is a covariant quantity.*

In other words, this says that then the covariant general coordinate transformation P_a gives also a covariant quantity.

The result of this operation depends on whether the algebra is closed on the original field. Thus we consider a field ϕ with transformation law and commutator

$$\delta_I(\epsilon^I)\phi = \epsilon^I\chi_I\,, \qquad \left[\delta_I(\epsilon_1^I),\delta_J(\epsilon_2^J)\right]\phi = \epsilon_2^J\epsilon_1^I\left(f_{IJ}{}^K\chi_K + f_{IJ}{}^a D_a\phi + \eta_{IJ}\right)\,. \tag{3.20}$$

The last tensor, η_{IJ} is the non-closure function. If $\eta_{IJ}=0$, then the 'algebra closes', and $D_a\phi$ will transform itself in another covariant quantity. If the algebra does not close, then $D_a\phi$ is still a covariant quantity, but its transformation is not.

The same holds for curvatures, which are in fact the generalisation of the covariant derivatives for gauge fields:

Theorem on covariant curvatures. *For a gauge field with transformation law as in (3.8) where $M_{aJ}{}^I$ is a covariant quantity, the covariant curvature (3.19), with (3.12) is a covariant quantity.*

I will now first give a proof of the theorem of covariant derivatives, and then discuss an example of the vector multiplet in 6 dimensions. I will show in there how the covariant general coordinate indeed needs the seemingly non-covariant last term in (3.11), but how the theorem on the covariant curvature is realized. $\widehat{R}_{\mu\nu}$ does not transform to a covariant quantity, but $\widehat{R}_{ab}$ does. These are very difficult calculations, and should show you why these theorems are so useful, i.e. to avoid that you have to do such manipulations explicitly. Finally, I will show in examples the 'easy way' in which to get results of transformations due to the theorems.

3.2.1 Proof of theorem on covariant derivatives

This is just some formal calculation, starting from (3.20) and (3.8)

$$\begin{aligned}\delta_J(\epsilon^J)D_\mu\phi &= \epsilon^I\partial_\mu\chi_I - \epsilon^J h_\mu^K f_{KJ}{}^I\chi_I - \epsilon^J\mathcal{M}_{\mu J}{}^I\chi_I - h_\mu^I\delta_J(\epsilon^J)\chi_I \\ &= \epsilon^I\left(D_\mu+\delta_J(h_\mu^J)\right)\chi_I - \epsilon^J h_\mu^K f_{KJ}{}^I\chi_I - \epsilon^J\mathcal{M}_{\mu J}{}^I\chi_I - \delta_J(\epsilon^J)\delta_I(h_\mu^I)\phi\,.\end{aligned}\tag{3.21}$$

For the last term, I used the notation in the way it is used in calculating a commutator. That means, that the first $\delta(\epsilon)$ does not act on the h_μ^I within the other transformation in the same way as it is done when one calculates a commutator. Then the other term of the commutator comes from the term that was introduced to make the difference between ∂_μ

and D_μ. One gets thus a commutator with ϵ_1^I replaced by h_μ^I and ϵ_2^I by ϵ^I. Using then (3.20), I obtain

$$\begin{aligned}\delta_J(\epsilon^J)D_\mu\phi &= \epsilon^I D_\mu\chi_I - \epsilon^J\mathcal{M}_{\mu J}{}^I\chi_I + \epsilon^J h_\mu^I\left(f_{IJ}{}^a D_a\phi + \eta_{IJ}\right)\\ &= \epsilon^I D_\mu\chi_I - \epsilon^J\mathcal{M}_{\mu J}{}^I\chi_I + \epsilon^J h_\mu^I\eta_{IJ} + \left(\delta_I(\epsilon^I)e_\mu{}^a - \epsilon^I e_\mu^b f_{bI}{}^a\right) D_a\phi\,. \quad (3.22)\end{aligned}$$

At the end, I made use of (3.7). The first term in the bracket has an explicit ψ_μ, but is cancelled when reverting to the transformation of $D_a\phi$:

$$\delta_J(\epsilon^J)D_a\phi = \epsilon^I D_a\chi_I - \epsilon^J\mathcal{M}_{aJ}{}^I\chi_I + \epsilon^J h_a^I\eta_{IJ} - \epsilon^I f_{aI}{}^b D_b\phi\,. \quad (3.23)$$

If $\eta_{IJ} = 0$, the transformation of the derivative is again a covariant quantity. I will come back to the meaning of the last term in section 3.2.4.

3.2.2 Example of $d = 6$ vector multiplet

I will use an example to illustrate first the closure of the algebra on gauge fields. In the next part, section 3.2.3 it is used to show how the transformation of curvatures gives a covariant result. Finally, in section 3.2.4 it is used to illustrate how to make calculations easily using the theorems.

The example, the $d = 6$ vector multiplet, consists of a vector W_μ and a spinor Ω^i. The transformation laws excluding covariant general coordinate transformations are:

$$\begin{aligned}\delta_I(\epsilon^I)W_\mu &= \partial_\mu\alpha - \bar\epsilon\gamma_\mu\Omega\,,\\ \delta_I(\epsilon^I)\Omega^i &= \left(\tfrac32\Lambda_D - \tfrac14\gamma^{ab}\lambda_{ab}\right)\Omega^i + \Lambda^{ij}\Omega_j + \tfrac18\gamma^{ab}\widehat{F}_{ab}\epsilon^i\,. \quad (3.24)\end{aligned}$$

α is the parameter for the U(1) transformation that W_μ gauges. This U(1) commutes with all other symmetries. W_μ is thus one of the h_μ^I in the general treatment, and comparing with (3.8), one can identify

$$M_{\mu(\alpha i)}{}^{U(1)} = -\left(\gamma_\mu\Omega_i\right)_\alpha\,. \quad (3.25)$$

(αi) stands here for the combined index indicating a supersymmetry. The i-index is implicit in the first line of (3.24), as explained in (A.40). This implies that the modified curvature (which is called $\widehat{F}_{\mu\nu}$ here) is

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + 2\bar\psi_{[\mu}\gamma_{\nu]}\Omega, \qquad F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}. \quad (3.26)$$

Instead of using (3.12), just recognize that derivatives in $F_{\mu\nu}$ have to be completed to covariant derivatives.

In the second line of (3.24), the Λ_D-term states that Ω has 'Weyl weight' $\frac32$. I will explain in section 4.2 how this weight can be easily obtained. The Lorentz transformation is general for all the spinors. Its form can in fact already be seen from the commutators in (2.51). Similarly, the SU(2) transformation is general for any doublet. Finally, observe that the supersymmetry transformation involves $\widehat{F}_{ab}$. This should be such because Ω is a matter field, and should thus transform to a covariant quantity. In section 3.2.3, I will explain how

$\widehat{F}_{ab}$ is indeed a covariant quantity. But first, I want to comment on the covariant general coordinate transformations, for which this expression is also necessary.

Following the formulae for covariant general coordinate transformations, one finds from (3.11)

$$\begin{aligned}\delta_{\rm cgct}(\xi)W_\mu &= -\xi^\nu \widehat{F}_{\mu\nu} + \xi^a\bar\psi_\mu\gamma_a\Omega\\ \delta_{\rm cgct}(\xi)\Omega^i &= \xi^a D_a\Omega^i\,,\qquad D_\mu\Omega^i = \mathcal{D}_\mu\Omega^i - \tfrac18\bar\psi^i_\mu\gamma^{ab}\widehat{F}_{ab}\,,\\ \mathcal{D}_\mu\Omega^i &= \left(\partial_\mu - \tfrac32 b_\mu + \tfrac14\gamma^{ab}\omega_{\mu ab}\right)\Omega^i - V_\mu{}^{ij}\Omega_j\,.\end{aligned} \tag{3.27}$$

I first comment on the last expression. I introduce here also a $\mathcal{D}$ for convenience, that only includes the D, M_{ab} and R-symmetry, i.e. the linearly realized symmetries. That is convenient as they have a general form, corresponding to the explanations above of the form of these symmetries. They are also easy to calculate with, as will be illustrated below.

The last term in the first line may look strange, but is necessary for a correct result of the anticommutator of two supersymmetries. Indeed, consider the exercise of calculating the commutator of two supersymmetries on W_μ. That should now give a covariant general coordinate transformation.

$$\delta_Q(\epsilon_1)\delta_Q(\epsilon_2)W_\mu = -\bar\epsilon_2\gamma_\mu\delta_Q(\epsilon_1)\Omega - \bar\epsilon_2\gamma_a\Omega\,\delta_Q(\epsilon_1)e_\mu{}^a\,. \tag{3.28}$$

The first term gives clearly the covariant curvature, and the second one leads to the second term in the last expression of (3.11).

Exercise 3.2: Check further that one obtains indeed the right coefficient for the transformations, using (2.6), symmetries using (A.54), Fierz formulae (A.66), gamma manipulations using (A.60), and (A.18).

3.2.3 Illustration of full calculation of the transformation of a curvature

I will now show that $\widehat{F}_{ab}$ is a covariant quantity. I first want to calculate the supersymmetry transformation of $\widehat{F}_{\mu\nu}$. Therefore I need $\delta_Q\psi_\mu$. The gauge algebra determines several terms, and there may be matter terms that we have not discussed yet, and so I leave them arbitrary, calling them Υ_μ:

$$\delta_Q\psi_\mu = \mathcal{D}_\mu\epsilon + \Upsilon_\mu\,,\qquad \mathcal{D}_\mu\epsilon^i \equiv \left(\partial_\mu + \tfrac12 b_\mu + \tfrac14\gamma^{ab}\omega_{\mu ab}\right)\epsilon^i - V_\mu{}^{ij}\epsilon_j\,. \tag{3.29}$$

Therefore we have

$$\delta_Q(\epsilon)\widehat{F}_{\mu\nu} = -2\partial_{[\mu}\left(\bar\epsilon\gamma_{\nu]}\Omega\right) + 2\overline{(\mathcal{D}_{[\mu}\epsilon)}\gamma_{\nu]}\Omega + 2\bar\Upsilon_{[\mu}\gamma_{\nu]}\Omega + 2\bar\psi_{[\mu}\delta_Q(\epsilon)\left(\gamma_{\nu]}\Omega\right)\,. \tag{3.30}$$

The first ∂_μ can be replaced by a curly covariant derivative $\mathcal{D}_\mu$, as it acts on a SU(2) scalar, Lorentz scalar, and D-invariant quantity. To say so, I consider for now the parameter ϵ as an SU(2) doublet, Lorentz spinor, and of dilatational weight $-\frac12$ (check that Ω got weight $\frac32$ and the implicit $e_\nu{}^a$ has weight -1, see (3.7)). That is implicit in the definition of $\mathcal{D}_\mu\epsilon^i$ in

(3.29). Then we can 'distribute' this covariant derivative, and terms with $\mathcal{D}_\mu$ on ϵ cancel. This illustrates the convenience to work with the $\mathcal{D}_\mu$ derivatives. I am left with

$$\delta_Q(\epsilon)\widehat{F}_{\mu\nu} = -2\bar{\epsilon}\gamma_{[\nu}\mathcal{D}_{\mu]}\Omega - 2\bar{\epsilon}\gamma_a\Omega\,\mathcal{D}_{[\mu}e_{\nu]}{}^a + 2\bar{\Upsilon}_{[\mu}\gamma_{\nu]}\Omega - 2\delta_Q(\epsilon)\delta_Q(\psi_{[\mu})W_{\nu]}\,. \tag{3.31}$$

The writing of the last term is similar to what was done in section(3.2.1), and the following manipulations are similar to those that I did there. Replacing the first curly derivative by a full covariant derivative, this can be written as $\mathcal{D}_\mu\Omega = D_\mu\Omega + \delta_Q(\psi_\mu)\Omega$. The latter term nearly leads to $\delta_Q(\psi_\mu)\delta_Q(\epsilon)W_\nu$, apart from that we have to be careful that also the vierbein transforms in the latter expression. We obtain:

$$\begin{aligned}\delta_Q(\epsilon)\widehat{F}_{\mu\nu} \;=\;& -2\bar{\epsilon}\gamma_{[\nu}D_{\mu]}\Omega - 2\bar{\epsilon}\gamma_a\Omega\,\mathcal{D}_{[\mu}e_{\nu]}{}^a + 2\bar{\Upsilon}_{[\mu}\gamma_{\nu]}\Omega \\ &+2\left[\delta_Q(\psi_{[\mu}),\delta_Q(\epsilon)\right]W_{\nu]} + 2\bar{\epsilon}\gamma_a\Omega\delta_Q(\psi_{[\mu})e_{\nu]}{}^a\,.\end{aligned} \tag{3.32}$$

We already calculated the commutator on W_μ, checking that it gives the covariant general coordinate transformation. The parameter that we have to use here is, see (2.6)

$$\xi^a(\psi_\mu,\epsilon) = \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu = \delta_Q(\epsilon)e_\mu{}^a\,. \tag{3.33}$$

That can be inserted in (3.27) and one can use the constraint (3.5) to obtain

$$0 = 2\mathcal{D}_{[\mu}e_{\nu]}{}^a + \xi^a(\psi_\mu,\psi_\nu)\,. \tag{3.34}$$

I leave to you that after further Fierz manipulations as in exercise 3.2, one arrives at

$$\delta_Q(\epsilon)\widehat{F}_{\mu\nu} = -2\bar{\epsilon}\gamma_{[\nu}D_{\mu]}\Omega + 2\bar{\Upsilon}_{[\mu}\gamma_{\nu]}\Omega - 2\widehat{F}_{a[\nu}\delta_Q(\epsilon)e_{\mu]}{}^a\,. \tag{3.35}$$

The last term contains an explicit ψ_μ. It is clear that this is cancelled when calculating the transformation

$$\delta_Q(\epsilon)\widehat{F}_{ab} = -2\bar{\epsilon}\gamma_{[b}D_{a]}\Omega + 2\bar{\Upsilon}_{[a}\gamma_{b]}\Omega\,. \tag{3.36}$$

This does not contain any explicit gauge fields. The gauge fields are hidden in the covariant derivative.

3.2.4 The easy way

This was to show that it works. That was complicated, and I was only looking to a simple example! However, now comes the good news: once you know the tricks, you never have to do it. The tricks involve a good use of calculus with covariant quantities. The tricks involve the knowledge of the following facts:

- D_a on a covariant quantity is a covariant quantity, and so is $\widehat{R}_{ab}$.
- The transformation of a covariant quantity does not involve a derivative of a parameter.
- If the algebra closes on the fields, then the transformation of a covariant quantity is a covariant quantity, i.e. gauge fields only appear either included in covariant derivatives or in curvatures.

Let us first consider again the calculation of $\delta\widehat{F}_{ab}$. The principles imply that this should be a covariant quantity. First, I consider the definition (3.26). For the transformation of $F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}$, I consider the transformation of W_ν, and when taking the ∂_μ derivative, I delete the term where the derivative acts on the parameter, because this has to disappear in the transformation of a covariant quantity. I also do not have to consider derivatives on a vielbein. A derivative on any gauge field can only appear as its curvature, but we know that the curvature for the vierbein has been constrained to zero. So from the variation of the first term in (3.26) remains only the first term of (3.36), modulo terms that should at the end disappear. In the transformation of the second term, I should consider only

$$\left(\delta_Q\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\Omega\,. \tag{3.37}$$

Indeed, if we would act with δ_Q on the other factors, then a ψ_μ remains and we know that these should cancel anyway in the full result. Moreover for the variation of ψ_μ, as for any h_μ^I gauge field, one can neglect the first and second terms in the second line of (3.8) for the reasons already mentioned. Thus, the only relevant part of the transformation is $\mathcal{M}$, which is in our case the Υ-term.

Exercise 3.3: Check that, whatever would be the S-transformation of Ω (here in 6 dimensions it is zero, however the corresponding fermion in 4 and 5 dimensions has S-supersymmetry transformations), the S-variation of $\widehat{F}_{ab}$ is due to (3.10):

$$\delta_S(\eta)\widehat{F}_{ab} = -2\bar{\eta}\gamma_{ab}\Omega\,. \tag{3.38}$$

Exercise 3.4: One can even give a general formula for the transformation of curvatures, correcting (2.61) for the effects that gauge fields transform with matter-like terms. To apply the methods explained earlier, the last decomposition in (3.12) is most useful. Indeed, explicit gauge fields appear only quadratically in ${r_{ab}}^I$. You can then determine that

$$\delta_J(\epsilon^J)\widehat{R}_{ab}{}^I = \epsilon^J\widehat{R}_{ab}{}^K f_{KJ}{}^I + 2\epsilon^J D_{[a}\mathcal{M}_{b]J}{}^I - 2\epsilon^K\mathcal{M}_{[aK}{}^J\mathcal{M}_{b]J}{}^I\,. \tag{3.39}$$

Similarly, the Bianchi identity becomes

$$D_{[a}\widehat{R}_{bc]}{}^I - 2\widehat{R}_{[ab}{}^J\mathcal{M}_{c]J}{}^I = 0\,. \tag{3.40}$$

Let me now consider again the *transformation of a covariant derivative* on a covariant quantity. The exact calculation lead to (3.23).

The first term is the covariantization of the leading term in the transformation of $\partial_\mu\phi$.

The second term, is then the one that arises from the transformation of the gauge fields in the second term of the definition (3.16). Again, I only have to consider the transformation of the gauge field itself in that term, as other terms would remain with an explicit gauge term. And moreover in the transformation of the gauge fields, I only have to consider the terms $\mathcal{M}$. These terms are thus from a practical point of view the most interesting ones. They are of two types, see (3.8). The first ones are those from the gauge algebra where the gauge field was the vielbein. These are the transformations explicitly given in (3.10). The others are the matter terms, which we still have to find for each case. For the gauge field in the vector multiplet, that was the relevant term, which gave its supersymmetry transformation to the gaugino.

The third term is only in case of non-closure. I will consider below the fermionic field in the vector multiplet for which there is no closure. Closure could be obtained if I would have introduced auxiliary fields, but for the didactical value of the example, it is good to consider the situation without the auxiliary fields.

First, consider the fourth term, that finds its origin in transformations of the vielbein to the vielbein. These are the first two terms in the explicit expression of (3.7). These two terms amount to the following. First it implies that the Weyl weight of $D_a\phi$ is one higher than that of ϕ. Second, it implies that the Lorentz transformation of $D_a\phi$ differs from the one of ϕ corresponding to the structure with one extra Lorentz index.

So, finally, consider the gaugino Ω^i in the $d = 6$ vector multiplet. As we have already calculated the supersymmetry transformation of $\widehat{F}_{ab}$, it is easy to calculate the commutator of two supersymmetries on the gaugino. I do not take the Υ-term into account, as we do not know its form yet, and it is independent of the rest. The commutator is

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]\,\Omega^i = -\tfrac{1}{2}D_c\Omega^i\,\bar\epsilon_1\gamma^c\epsilon_2 + \tfrac{3}{16}\gamma_c\,\not{D}\Omega^i\,\bar\epsilon_1\gamma^c\epsilon_2 + \tfrac{1}{96}\gamma_{cde}\,\not{D}\Omega_j\,\bar\epsilon_1^{(i}\gamma^{cde}\epsilon_2^{j)}\,. \qquad (3.41)$$

The first term is the covariant general coordinate transformation. The others are the non-closure terms. Remark that they are proportional to $\not{D}\Omega$ These imply that the supersymmetry variation of $D_a\Omega$ is (neglecting again possible extra matter terms in transformations of gauge fields)

$$\delta_Q(\epsilon)D_a\Omega^i = \tfrac{1}{8}\gamma^{bc}\epsilon^i D_a\widehat{F}_{bc} + \tfrac{3}{16}\gamma_c\,\not{D}\Omega^i\,\bar\psi_a\gamma^c\epsilon + \tfrac{1}{96}\gamma_{cde}\,\not{D}\Omega_j\,\bar\psi_a^{(i}\gamma^{cde}\epsilon^{j)}\,. \qquad (3.42)$$

Thus, in this case the transformation of (the covariant) $D_a\Omega$ is not a covariant quantity. It is now not possible to define $D_bD_a\Omega$ such that it does not transform to a derivative of a gauge field. However, note that one can define it such that at least the antisymmetric part in $[ab]$ does not transform in a derivative. That is analogous to a curvature. Also on a gauge field we cannot define a covariant generalisation of $\partial_{(\mu}W_{\nu)}$. The covariant $D_{[b}D_{a]}$ should just have extra factors $\frac{1}{2}$ for every term in which there are two gauge fields. Also that is similar to a curvature:

$$D_{[b}D_{a]}\Omega^i = \partial_{[b}D_{a]}\Omega^i - \tfrac{1}{8}\gamma^{cd}\psi^i_{[b}D_{a]}\widehat{F}_{cd} - \tfrac{3}{32}\gamma_c\,\not{D}\Omega^i\,\bar\psi_a\gamma^c\psi_b - \tfrac{1}{192}\gamma_{cde}\,\not{D}\Omega_j\,\bar\psi_a^{(i}\gamma^{cde}\psi_b^{j)}\,. \qquad (3.43)$$

But these are objects that one seldom needs.

3.3 Curvature constraints and their consequences

3.3.1 Consequences of the constraint on $R(P)$

I mentioned already one constraint: (3.5), the vanishing of the P^a curvature. You see immediately why this constraint can be imposed. Concerning representation content, it is a vector times an antisymmetric tensor, and the same holds for the spin connection $\omega_\mu{}^{ab}$. So, one can solve it for this spin connection. That is why the constraint is called a 'conventional constraint'. Moreover, that is what I was looking for, see the first item of the list of three shortcomings in the beginning of this chapter. So, from now on, we have

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + 2e_\mu{}^{[a}b^{b]} - \xi^{[a}(\psi_\mu, \psi^{b]}) - \tfrac{1}{2}\xi_\mu(\psi^a, \psi^b)\,, \tag{3.44}$$

where $\omega_\mu{}^{ab}(e)$ is the usual expression of the spin connection, see (A.4), obtained by the antisymmetric part of (A.3), which is a simplification of the above constraint without b_μ and gravitinos. The second term can be understood from the special conformal invariance, see (3.10). The third contribution is already known from pure supergravity [33, 34, 35].

In fact, let me make a remark that, so far, the whole discussion is not specific for conformal supergravity. All the expressions would hold also for gauging Poincaré supergravity (apart from the fact that you should delete the b_μ-terms, ...).

The constraint is not invariant under all the symmetries. Consider its transformation, using my 'easy method'. The terms quadratic in the gravitino do not play a role in the variation, as they will always leave a naked gravitino behind. Thus we should only consider the variations of the vielbein in the derivative. According to (3.7), this involves only ∂e, i.e. $R_{ab}(P)$, and $\partial\psi$, i.e. $\widehat{R}_{ab}(Q)$. The former are zero, so there remains only a variation in $\widehat{R}_{ab}(Q)$:

$$\delta_I(\epsilon^I)R_{ab}(P^c) = \xi^c\left(\widehat{R}_{ab}(Q), \epsilon\right)\,. \tag{3.45}$$

One may think of two solutions. The first one is to put $\widehat{R}_{ab}(Q) = 0$. However, in the expression for $\widehat{R}_{ab}(Q)$ there is no field that is a spinor–antisymmetric tensor. Thus this would not be a conventional constraint. It would already impose differential constraints on fields, which is not what I want at this point. The second solution is easier. As $\omega_\mu{}^{ab}$ is now defined by the vanishing of $R_{ab}(P)$, its transformations can be defined by its expression in terms of the other fields. Another way to say so, is that $\delta\omega_\mu{}^{ab}$ has extra terms to compensate for the previous noninvariance of $R_{ab}(P)$. The result has the same structure as the solution (3.44):

$$\delta_{extra}\omega_\mu{}^{ab} = \xi^{[a}(\widehat{R}_\mu{}^{b]}(Q), \epsilon) + \tfrac{1}{2}\xi_\mu(\widehat{R}^{ab}(Q), \epsilon)\,. \tag{3.46}$$

This has to be seen as the additional M-term in (3.8).

A final consequence of the constraint can be seen from the Bianchi identity $\nabla_{[\mu}R_{\nu\rho]}(P^a)$. This equation contains terms with $R_{\mu\nu}(P^a)$, which are thus vanishing now, but also some covariantization terms that involve other curvatures multiplied with gauge fields. Some of these terms involve the gravitino, but in other terms the gauge field is the vielbein. E.g. there is the term $e^a_{[\mu}\widehat{R}_{\nu\rho]}(D)$. Once, I change all indices to tangent spacetime indices, then this is a relation between covariant quantities, e.g. $\widehat{R}_{ab}(D)$ and terms with gauge fields. But then all explicit gauge fields should disappear when I write everything in terms of covariant quantities. Therefore, finally I only have to consider the terms where the gauge field is the vielbein, and covariantize the curvatures. The only terms are then (Bianchi identity for $R(P^d)$, with $[\mu\nu\rho]$ turned to $[abc]$)

$$-\,\delta_{[a}{}^d\widehat{R}_{bc]}(D) + \widehat{R}_{[ab}(M_{c]}{}^d) = 0\,. \tag{3.47}$$

So far, this is independent of supersymmetry or even of the conformal group. In fact, the last identity, without $R(D)$ would then be the well-known equation that the M-curvature antisymmetrized in 3 indices is zero. Multiplying (3.47) with $\delta_d{}^a$ gives that the contracted M-curvature, i.e. the Ricci tensor, is symmetric if there is no $R_{ab}(D)$:

$$(2-d)\widehat{R}_{ab}(D) = 2\widehat{R}_{d[a}(M_{b]}{}^d)\,. \tag{3.48}$$

Exercise 3.5: The M-curvature can be calculated directly by taking the derivative of (3.44). As the final result has to be covariant, we can drop all other terms, and restrict to the $\partial\omega$ term, where the derivative acts either on $\omega(e)$ or on b_μ. This leads to a direct relation expressing $\widehat{R}_{\mu\nu}(M^{ab})$ in the covariantization of $\partial\omega(e)$, and $\widehat{R}_{\mu\nu}$. Check that in this terminology, (3.47) implies that the former part satisfies the cyclic identity $R_{[\mu\nu\rho]}{}^\sigma = 0$ and leads to a symmetric Ricci tensor.

Extra comment. One can define the affine connection $\Gamma^\rho_{\mu\nu}$ by a similar constraint as (3.5), but now symmetric in $(\mu\nu)$:

$$\left(\partial_{(\mu} + b_{(\mu}\right) e^a_{\nu)} - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega_{[\mu}{}^{ab} e_{\nu]b} = 0\,. \tag{3.49}$$

Then one has a covariantly constant vielbein in the sense that

$$D_\mu e^a_\nu = \left(\partial_\mu + b_\mu\right) e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega_\mu{}^{ab} e_{\nu b} + \tfrac{1}{2}\xi^a(\psi_\mu, \psi_\nu) = 0\,. \tag{3.50}$$

Including the affine connection $\Gamma^\rho_{\mu\nu}$ in covariant derivatives, we can use these covariant derivatives also on objects with local indices $\mu, \nu, \ldots$.

3.3.2 Other constraints

Now that we have seen the procedure how $\omega_\mu{}^{ab}$ became dependent, you may consider whether this can be done for other gauge fields in the same way. If we look for the smallest possible multiplet, then we should use this procedure for all the gauge fields that we can algebraically solve. The clue is given in (3.14). Indeed, that equation shows which gauge fields appear 'linearly' in the curvatures, in the sense that they are multiplied by a vielbein. This shows that we may choose constraints that determine $f_\mu{}^a$ and ϕ_μ. This looks rather convenient. Indeed, $f_\mu{}^a$ would without such a constraint already appear in bosonic conformal gravity, while we do not know this field in the physical conformal gravity. At first it looks that we can define it either from a constraint on the dilatational or on the Lorentz curvature. But (3.48) implies that $\widehat{R}(D)$ is a function of $\widehat{R}(M)$, so that we can restrict ourselves to the Lorentz curvature. To be able to eliminate $f_\mu{}^a$ completely, the constraint should be a general $d \times d$ matrix. Therefore the constraint should look like

$$\widehat{R}_{ac}(M^{bc}) + C_a{}^b = 0\,, \tag{3.51}$$

where $C_a{}^c$ may be any covariant function that is independent of $f_\mu{}^a$. Similarly, there can be a third constraint that determines ϕ_μ from a constraint of the form

$$\gamma^b \widehat{R}(Q)_{ba} + \rho_a = 0\,, \tag{3.52}$$

where ρ_a is a covariant spinor-vector.

I go now again through the consequences of these constraints. First, the purpose was to define the gauge fields. Let me repeat the names of the various covariantizations of curvatures (and introduce meanwhile another one: $\widehat{R}'$):

$$\begin{aligned}
r_{\mu\nu}{}^I &= 2\partial_{[\mu}h_{\nu]}{}^I + h_\nu{}^K h_\mu{}^J f_{JK}{}^I\,,\\
R^I_{\mu\nu} &= r_{\mu\nu}{}^I - 2h_{[\mu}{}^J f_{\nu]J}{}^I\,,\\
\widehat{R}_{\mu\nu}{}^I &= R_{\mu\nu}{}^I - 2h_{[\mu}{}^J M_{\nu]J}{}^I = r_{\mu\nu}{}^I - 2h_{[\mu}{}^J \mathcal{M}_{\nu]J}{}^I\,,\\
\widehat{R}'_{\mu\nu}{}^I &= r_{\mu\nu}{}^I - 2h_{[\mu}{}^J M_{\nu]J}{}^I = \widehat{R}_{\mu\nu}{}^I + 2h_{[\mu}{}^J f_{\nu]J}{}^I\,.
\end{aligned} \tag{3.53}$$

Furthermore, as a completion of the Ricci tensor, see (A.2), I define

$$\mathcal{R}_{\mu\nu} = \widehat{R}'_{\rho\mu}(M^{ab}) e_a{}^\rho e_{\nu b}\,, \qquad \mathcal{R} = \mathcal{R}_\mu{}^\mu\,. \tag{3.54}$$

The solution is

$$\begin{aligned}
2(d-2) f_\mu{}^a &= -\mathcal{R}_\mu{}^a - C_\mu{}^a + \frac{1}{2(d-1)} e_\mu{}^a \left(\mathcal{R} + C_b{}^b\right)\,,\\
s_d(d-2)\phi_a &= \gamma^b \widehat{R}'_{ab}(Q) - \rho_a - \frac{1}{2(d-1)}\gamma_a \left(\gamma^{bc}\widehat{R}'_{bc}(Q) - \gamma^b\rho_b\right)\,.
\end{aligned} \tag{3.55}$$

There are the same type of consequences as for the first constraint. The transformation law of the fields $f_\mu{}^a$ and ϕ_μ get extra matter-like contributions. I assume that the extra C and ρ terms in the constraints are taken with the same Lorentz structure, SU(2) structure, dilatational and U(1) weight as the curvature term. Then the constraints break only supersymmetry. In general, this can be as well Q as S supersymmetry. The exact rules depend on the transformations of the matter terms in the constraint. The simplicity of the result, e.g. invariance of the constraint under S-transformations may even be an argument for the choice of the matter terms in the constraints.

The K transformations of these constraints vanish due to the constraint on $R_{\mu\nu}(P)$, at least when the extra matter fields do not transform under K. Indeed, this will not be the case due to a Weyl-weight related argument that I will give in section 3.5.1.

Furthermore, there are relations between the curvatures. Those give relations for the S and K curvatures. The explicit form depends on the matter sector.

3.4 Conformal invariant actions

Let us now illustrate the construction of a conformal invariant action by looking at a scalar sigma model without supersymmetry. We have thus a manifold with scalars ϕ^i. The metric

g_{ij} defines the Levi-Civita connection Γ^k_{ij}. We assume that there is a closed homothetic Killing vector as in (2.30), which allows to define the conformal symmetry of the scalars. We have the gauge fields associated to the generators of the conformal group: $e_\mu{}^a$ associated to the translations P_a, $\omega_\mu{}^{ab}$ associated to Lorentz transformations M_{ab}, b_μ associated to the dilatations D, and finally $f_\mu{}^a$ associated to the special conformal transformations K_a. There are no other fields, and the transformation laws of the gauge fields under local dilatations (parameter λ_D), and special conformal transformations (parameter Λ_{Ka}) follow directly from (2.58):

$$\begin{aligned} \delta\phi^i &= k^i_D\lambda_D\,, \\ \delta e_\mu{}^a &= -\lambda_D e_\mu{}^a, \qquad \delta e^\mu{}_a = \lambda_D e^\mu{}_a, \\ \delta\omega_\mu{}^{ab} &= -4\Lambda_K^{[a} e_\mu^{b]}\,, \\ \delta b_\mu &= \partial_\mu\lambda_D + 2\Lambda_{K\mu} \\ \delta f_\mu{}^a &= \lambda_D f_\mu{}^a + \partial_\mu \Lambda_K{}^a\,, \end{aligned} \tag{3.56}$$

where as usual, $e^\mu{}_a$ is the inverse of $e_\mu{}^a$./

The covariant derivative for the scalars is thus

$$D_\mu\phi^i = \partial_\mu\phi^i - b_\mu k^i_D\,, \qquad D_a\phi^i = e^\mu_a D_\mu\phi^i\,. \tag{3.57}$$

This leads to

$$\delta D_a\phi^i = \lambda_D\left[D_a\phi^i + (\partial_j k^i_D)D_a\phi^j\right] - 2\Lambda_{Ka}k^i_D = \lambda_D\left[(w+1)D_a\phi^i - k^k_D\Gamma^i_{jk}D_a\phi^j\right] - 2\Lambda_{Ka}k^i_D\,. \tag{3.58}$$

The second covariant derivative is therefore

$$D_aD^a\phi^i = \partial_aD^a\phi^i + 2f^a_a k^i - b^a\left[(w+1)D_a\phi^i - k^k\Gamma^i_{jk}D_a\phi^j\right] + e^{\mu a}\omega_{\mu a}{}^b D_b\phi^i\,. \tag{3.59}$$

The covariant box is

$$\Box^c\phi^i = D^aD_a\phi^i + \Gamma^i_{jk}D_a\phi^j D^a\phi^k\,. \tag{3.60}$$

One checks

$$\delta_K\Box^c\phi^i = 2\Lambda^a_K(d-2-2w)D_a\phi^i\,. \tag{3.61}$$

For the action, one starts with

$$e^{-1}\mathcal{L} = \frac{1}{2}g_{ij}D_a\phi^iD^a\phi^j - \frac{1}{w}f^a_a g_{ij}k^i_Dk^j_D\,. \tag{3.62}$$

Both terms on the right-hand side scale under dilatations separately with a factor $2(w+1)$, while the left-hand side scales with weight d. So it is consistent under the same condition

$$w = \tfrac{1}{2}(d-2)\,, \tag{3.63}$$

which keeps the conformal box also K-invariant. The K transformations of the action give

$$e^{-1}\delta_K L = -2\Lambda^a_K k^i_D g_{ij}D_a\phi^j - \frac{1}{w}\left(\partial_a\Lambda^a_K\right)g_{ij}k^i_Dk^j_D = -\frac{1}{w}\partial_a\left(\Lambda^a_K g_{ij}k^i_Dk^j_D\right)\,, \tag{3.64}$$

where one uses that

$$\partial_a(g_{ij}k_D^i k_D^j) = 2wk_D^i g_{ij}\partial_a\phi^j\,. \qquad (3.65)$$

One also checks that

$$\frac{\delta}{\delta\phi^i}\int \mathrm{d}^d\mathcal{L} = -g_{ij}\Box^c\phi^j\,. \qquad (3.66)$$

There are no C-terms in the constraint that determines f_μ^a, and thus (3.55) simplifies to

$$f_a^a = -\frac{1}{4(d-1)}R\,. \qquad (3.67)$$

We now just write R, rather than $\mathcal{R}$ to indicate that there are no further corrections to the usual bosonic scalar curvature of the metric. Therefore, the Lagrangian (with negative kinetic terms for the scalars) is

$$e^{-1}\mathcal{L} = \frac{1}{2}g_{ij}D_a\phi^i D^a\phi^j + \frac{1}{2(d-1)(d-2)}Rg_{ij}k_D^i k_D^j\,. \qquad (3.68)$$

In flat space, $k^i = w\phi^i$ and we obtain

$$e^{-1}\mathcal{L} = \frac{1}{2}g_{ij}D_a\phi^i D^a\phi^j + \frac{d-2}{8(d-1)}Rg_{ij}\phi^i\phi^j\,. \qquad (3.69)$$

This is, for $d = 4$ the result that we started from in the example to show the strategy of the conformal method, see (2.10).

3.5 The Standard Weyl multiplets

So far, the third problem that was mentioned in the beginning of this chapter has not been considered. The number of bosonic and fermionic components of the independent gauge fields do not match. I will make the count below. The fact that the numbers do not match also implies that the supersymmetry algebra cannot give rise to invertible coordinate transformations. Indeed, so far I gave you general forms for the covariant general coordinate transformations and for the other gauge transformations, but nothing guarantees that supersymmetries anticommute to the covariant general coordinate transformations. The matter terms M in the transformations of the gauge fields should be chosen appropriately to obtain the right anticommutator. They should be functions of new matter fields. The general argument says that the algebra can only be satisfied if the number of bosonic and fermionic components match. (A detailed explanation of this theorem is in appendix B of [36].

3.5.1 Matter fields completing the Weyl multiplet

How do we have to count the number of components? The argument says that in a set of states on which the algebra $\{Q, Q\} = P$ is valid, with P invertible, there have to be an equal number of bosonic as fermionic states. E.g., in table 1, we considered '*on-shell counting*', i.e. only physical states were considered. When we further consider an 'on-shell representation', i.e. one for which the algebra works only when the equations of motion are satisfied, then the equality should hold for such states. But here I want more. I want the algebra to be realized without use of equations of motion. Indeed, the 'Weyl multiplet' that we are constructing should still be independent of equations of motion. It should be used for many different actions. Therefore I will consider '*off-shell counting*'. Thus I consider the algebra before using equations of motion.

To do so, one must realize that the algebra is not just $\{Q, Q\} = P$. In the anticommutator of supersymmetries appear also gauge transformations. We can restrict to $\{Q, Q\} = P$ if these can be deleted. Thus the argument about equal number of bosonic and fermionic states is only correct up to gauge transformations. Thus we should subtract the gauge degrees of freedom in order to perform on-shell counting[11].

So, I can start the counting. First consider the bosonic fields related to the conformal group. The spin connection and K-gauge fields are dependent, so I should not consider them. There remain thus the vielbein and b_μ, the gauge field of dilatations, that is $d^2 + d$ fields. But there are gauge degrees of freedom for the whole conformal group $\mathrm{SO}(2, d)$, i.e. $(d+2)(d+1)/2$. Therefore this leaves $d(d-1)/2-1$ degrees of freedom. For the R symmetry group. The gauge fields are $4d$ fields for $\mathrm{SU}(2)\times\mathrm{U}(1)$, i.e. 4 dimensions, and $3d$ for $d = 5$ or 6 when the gauge group is $\mathrm{SU}(2)$, however, the gauge invariance reduces it to $4(d-1) = 12$ for $d = 4$ and $3(d-1)$ for $d = 5, 6$. On the fermionic side, the gauge fields for S supersymmetry are dependent fields, so there are only the $8d$ gravitino components, which transform under Q and S supersymmetry, leaving $8(d-2)$ components. In any case, the count does not work (see table 3).

Exercise 3.6: Check that for $N = 1$ in 4 dimensions, the count would be $8 + 8$, thus one does not need any extra fields. You have to know that the R-symmetry group is then $\mathrm{U}(1)$.

A solution is presented in table 3. It involves an antisymmetric tensor T, with two indices in 4 dimensions (T_{ab}, you may take it antiselfdual, but then it is complex) [37], with two or three indices in 5 dimensions (these are dual to each other) [29], or an anti-self-dual real tensor in 6 dimensions [30]. Further there is a real scalar D and a fermion doublet χ^i.

The solution is not unique and the arguments to obtain it are not so obvious. The first one of this nature was obtained in 4 dimensions from splitting a linearized Poincaré multiplet [38, 39, 40] that was found before.

One way of constructing the set of fields is to make use of supercurrents. One considers a multiplet that has rigid superconformal symmetry, and considers the fields that couple to the Noether currents. This method has been used in various cases, see [5, 41, 42, 43, 44, 29]. We will not go into this subject any further here.

For the transformation laws, one can first consider all terms that are allowed from Lorentz structure, $\mathrm{SU}(2)$ structure, and most of all Weyl weight. The latter is a strong restriction, especially for the K and S transformations, which have a negative Weyl weight in the sense

[11] General coordinate transformations are also local gauge transformations, that are more general than fixed translations. Indeed, the argument on equal numbers of bosons holds already for rigid supersymmetry. So the general coordinate - equivalent states should also be subtracted.

Table 3: *Number of components in the fields of the standard Weyl multiplet. The columns $d = 4, 5, 6$ indicate the number of components of the fields in the first column with gauge transformations subtracted. The next columns contain the Weyl weight, the chiral weight (only for $\mathrm{d} = 4$, where it means that the transformation under* U(1) *is $\delta\phi = \mathrm{i}c\phi\Lambda_A$), and the chirality for the fermions for even dimension. Note that for $\mathrm{d} = 4$, changing the position of the index changes the chirality, while in 6 dimensions the chirality is generic. Finally, we indicate the symmetry for which it is a gauge field, and possibly other gauge transformations that have been used to reduce its number of degrees of freedom in this counting.*

	$d=4$	$d=5$	$d=6$	w	c	γ_5/γ_7	gauge transf.	subtracted
$e_\mu{}^a$	5	9	14	-1	0		P^a	M_{ab}, D
b_μ	compensating K^a			0	0		D	K^a
$\omega_\mu{}^{ab}$	composite			0	0		M^{ab}	
$f_\mu{}^a$	composite			1	0		K^a	
$V_{\mu i}{}^j$	9	12	15	0	0		SU(2)	
A_μ	3			0	0		U(1)	
$\psi_\mu{}^i$	16	24	32	$-\frac{1}{2}$	$-\frac{1}{2}$	$+$	Q^i	S^i
$\phi_\mu{}^i$	composite			$+\frac{1}{2}$	$-\frac{1}{2}$	$-$	S^i	
T_{ab}^-, T_{abc}^-	6	10	10	1	-1			
D	1	1	1	2	0			
χ^i	8	8	8	$\frac{3}{2}$	$-\frac{1}{2}$	$+$		
TOTAL	$24+24$	$32+32$	$40+40$					

that in order to be in accordance with the commutation relations (2.51), a transformation $\delta\phi = \chi_a \Lambda_K^a$ can only occur of χ_a has a Weyl weight that is 1 lower than that of ϕ (formally consider therefore Λ_K to be of Weyl weight 1). The transformations are supposed to be local. That implies that one can only encounter covariant derivatives on the fields that are in the table. We saw already in section 3.2.4 that the covariant derivative increases the Weyl weight with one. Curvatures have Weyl weight two higher than that of the corresponding gauge field (due to the fact that we have to change to indices $[ab]$). The covariant object of lowest Weyl weight is thus the antisymmetric tensor T. Due to the Lorentz structure it cannot appear in $\delta_K D$, the object with highest Weyl weight (the terms ${M_{aK}}^J$ should have Weyl weight one lower than the one of the corresponding gauge field).

Continuing the analysis this way, one finds all possible terms, and then one has to check the algebra to determine coefficients. One more subtlety that appears at this time is that the algebra may be modified from the pure group-theoretical one by structure functions depending on the matter fields. One only has to demand that the same commutators are realized on all fields.

The resulting algebra is called a ‘soft algebra’, meaning that the structure constants are replaced by structure functions XXX ref. Sohnius XXX. These structure functions depend on the fields of the Weyl multiplet.

The algebra that we obtain here should afterwards be imposed for all matter multiplets, apart from possibly additional transformations under which the fields of the Weyl multiplets do not transform, and possibly field equations if these matter multiplets are on-shell.

I present here the solutions with the fields mentioned in table 3, which are called the ‘Standard Weyl multiplets’. For 5 and 6 dimensions alternative versions for Weyl multiplets have been constructed [30, 29], which differ from the above ones in the choice of auxiliary fields (fields below the double line in table 3). These multiplets contain a dilaton auxiliary field and are therefore called the ’Dilaton Weyl multiplets’. The analogue in $d = 4$ is probably the ‘new’ multiplet that is mentioned in section 4.4 of [45].

3.5.2 $d = 4$

Here is the result for the Q and S supersymmetry transformations for $d = 4$. First for the independent gauge fields:

$$
\begin{aligned}
\delta_{Q,S}(\epsilon,\eta)e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon^i\gamma^a\psi_{\mu i} + \text{h.c.}\,,\\
\delta_{Q,S}(\epsilon,\eta)b_\mu &= \tfrac{1}{2}\bar\epsilon^i\phi_{\mu i} - \tfrac{1}{2}\bar\eta^i\psi_{\mu i}\\
&\quad -\tfrac{3}{8}\bar\epsilon^i\gamma_\mu\chi_i + \text{h.c.}\,,\\
\delta_{Q,S}(\epsilon,\eta)A_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \tfrac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i}\\
&\quad +\tfrac{3}{8}\mathrm{i}\bar\epsilon^i\gamma_\mu\chi_i + \text{h.c.}\,,\\
\delta_{Q,S}(\epsilon,\eta)\mathcal{V}_\mu{}_i{}^j &= -\bar\epsilon_i\phi_\mu^j - \bar\eta_i\psi_\mu^j\\
&\quad +\tfrac{3}{4}\bar\epsilon_i\gamma_\mu\chi^j - (\text{h.c.}\,;\,\text{traceless})\,,\\
\delta_{Q,S}(\epsilon,\eta)\psi_\mu^i &= \left(\partial_\mu + \tfrac{1}{2}b_\mu + \tfrac{1}{4}\gamma^{ab}\omega_{\mu ab} + \tfrac{1}{2}\mathrm{i}A_\mu\right)\epsilon^i + V_\mu{}^i{}_j\epsilon^j\\
&\quad -\tfrac{1}{16}\gamma\cdot T^-\varepsilon^{ij}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\
\delta_{Q,S}(\epsilon,\eta)T_{ab}^- &= 2\bar\epsilon^i\widehat{R}_{ab}(Q)^j\varepsilon_{ij}\,,\\
\delta_{Q,S}(\epsilon,\eta)\chi^i &= -\tfrac{1}{24}\gamma\cdot\not{D}T^-\varepsilon^{ij}\epsilon_j - \tfrac{1}{6}\widehat{R}_j{}^i\cdot\gamma\epsilon^j - \tfrac{1}{6}\mathrm{i}\widehat{R}(A)\cdot\gamma\epsilon^i + \tfrac{1}{2}D\,\epsilon^i + \tfrac{1}{12}\gamma\cdot T^-\varepsilon^{ij}\eta_j\,,\\
\delta_{Q,S}(\epsilon,\eta)D &= \tfrac{1}{2}\bar\epsilon^i\not{D}\chi_i + \text{h.c.}\,,
\end{aligned}
\tag{3.70}
$$

where $\widehat{R}_j{}^i$ is the SU(2) curvature, $R(A)$ the U(1) curvature, and $\gamma\cdot R = \gamma^{ab}R_{ab}$. The notation $A^i{}_j - (\text{h.c.}\,;\,\text{traceless})$ stands for $A^i{}_j - A_j{}^i - \frac{1}{2}\delta_j{}^i(A^k{}_k - A_k{}^k)$. For the gauge fields, the first line represents the original gauge transformations, and the second line are the terms that were symbolically represented by $\mathcal{M}$. Observe the covariance of these terms and the transformations of the matter terms.

The dependent fields are defined by the constraints

$$
\begin{aligned}
0 &= R_{\mu\nu}^a(P)\,,\\
0 &= \gamma^b\widehat{R}_{ba}(Q)^i + \tfrac{3}{2}\gamma_a\chi^i\,,\\
0 &= \widehat{R}_{ac}(M^{bc}) + \mathrm{i}\widehat{\widetilde{R}}_a{}^b(A) + \tfrac{1}{4}T_{ca}^-T^{+bc} + \tfrac{3}{2}\delta_a{}^bD\,.
\end{aligned}
\tag{3.71}
$$

The terms that could appear in these constraints are fixed by compatibility with Weyl weights (thus that we do not want to modify the dilatational transformations), and the coefficients are chosen such that they are invariant under S-supersymmetry, avoiding extra S-transformations for the constrained fields. Observe, however, that this is a choice for convenience, which is even not necessary possible (it is not possible in $d = 5$ [29] or for $(2,0)$ in $d = 6$ [44]). In principle all constraints of the form (3.51) and (3.52) are equivalent up to field redefinitions. I present the transformations of the constrained fields in 3 lines: first the original transformations, then the M-transformations determined by the non-invariance of the constraints, and finally terms that represent modified structure functions:

$$\begin{aligned}
\delta_{Q,S}(\epsilon,\eta)\omega_\mu{}^{ab} &= \tfrac12\bar\epsilon^i\gamma^{ab}\phi_{\mu i}+\tfrac12\bar\eta^i\gamma^{ab}\psi_{\mu i}\\
&\quad -\tfrac38\bar\epsilon^i\gamma_\mu\gamma^{ab}\chi_i-\tfrac12\bar\epsilon^i\gamma_\mu\widehat{R}^{ab}(Q)_i\\
&\quad -\tfrac14\bar\epsilon^iT^{+ab}\varepsilon_{ij}\psi_\mu^j+\text{h.c.}\,,\\
\delta_{Q,S}(\epsilon,\eta)f_\mu^a &= \tfrac12\bar\eta^i\gamma^a\phi_{\mu i}\\
&\quad -\tfrac3{16}e_\mu{}^a\bar\epsilon^i\not{D}\chi_i+\tfrac14\bar\epsilon^i\gamma_\mu D_b\widehat{R}^{ba}(Q)_i\\
&\quad -\tfrac18\bar\epsilon^i\psi_\mu^j\,D_bT^{+ba}\varepsilon_{ij}-\tfrac3{16}\bar\epsilon^i\gamma^a\psi_{\mu i}\,D+\text{h.c.}\,,\\
\delta_{Q,S}(\epsilon,\eta)\phi_\mu^i &= \left(\partial_\mu-\tfrac12 b_\mu+\tfrac14\gamma^{ab}\omega_{\mu ab}+\tfrac12\mathrm{i}A_\mu\right)\eta^i+V_\mu{}^i{}_j\eta^j-f_\mu^a\gamma_a\epsilon^i\\
&\quad -\tfrac1{32}\not{D}T^-\cdot\gamma\gamma_\mu\varepsilon^{ij}\epsilon_j-\tfrac18\widehat{R}_j{}^i\cdot\gamma\gamma_\mu\epsilon^j+\tfrac18\mathrm{i}\widehat{R}(A)\cdot\gamma\gamma_\mu\epsilon^i\\
&\quad +\tfrac38\left[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu^i-(\bar\chi_j\gamma^a\psi_\mu^j)\gamma_a\epsilon^i\right].
\end{aligned}\tag{3.72}$$

These third lines thus show that the commutator between two supersymmetries got modified. The new commutator is

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]=\delta_{\rm cgct}\left(\xi_3^a\right)+\delta_M\left(\lambda_3^{ab}\right)+\delta_K\left(\Lambda_{3K}^a\right)+\delta_S\left(\eta_3^i\right)\,,\tag{3.73}$$

where the associated parameters are given by the following expressions:

$$\begin{aligned}
\xi_3^a &= \tfrac12\bar\epsilon_2^i\gamma^a\epsilon_{1i}+\text{h.c.}\,,\\
\lambda_3^{ab} &= \tfrac14\bar\epsilon_1^i\epsilon_2^j\,T^{+ab}\varepsilon_{ij}+\text{h.c.}\,,\\
\Lambda_{3K}^a &= \tfrac18\bar\epsilon_1^i\epsilon_2^j\,D_bT^{+ba}\varepsilon_{ij}+\tfrac3{16}\bar\epsilon_2^i\gamma^a\epsilon_{1i}\,D+\text{h.c.}\,,\\
\eta_3^i &= \tfrac34\,\bar\epsilon^i_{[1}\epsilon^j_{2]}\,\chi_j\,.
\end{aligned}\tag{3.74}$$

These extra terms can become important in applications where they can give rise to central charges if the fields appearing in the structure functions get non-zero vacuum expectation values. We will see that in the presence of vector multiplets, there appear extra terms of a similar nature involving the matter scalars of the vector multiplet.

3.5.3 $d = 5$

The Q- and S-supersymmetry transformation laws of the independent fields using again the same splitting in two lines for gauge terms and matter terms[12]

$$
\begin{aligned}
\delta_{Q,S}(\epsilon,\eta)e_\mu{}^a &= \tfrac12\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta_{Q,S}(\epsilon,\eta)b_\mu &= \tfrac12\mathrm{i}\bar\epsilon\phi_\mu+\tfrac12\mathrm{i}\bar\eta\psi_\mu\\
&\quad -2\bar\epsilon\gamma_\mu\chi\,,\\
\delta_{Q,S}(\epsilon,\eta)V_\mu{}^{ij} &= -\tfrac32\mathrm{i}\bar\epsilon^{(i}\phi_\mu^{j)}+\tfrac32\mathrm{i}\bar\eta^{(i}\psi_\mu^{j)}\\
&\quad +4\bar\epsilon^{(i}\gamma_\mu\chi^{j)}+\mathrm{i}\bar\epsilon^{(i}\gamma^{ab}T_{ab}\psi_\mu^{j)}\,,\\
\delta_{Q,S}(\epsilon,\eta)\psi_\mu^i &= \partial_\mu\epsilon^i+\tfrac12 b_\mu\epsilon^i+\tfrac14\omega_\mu^{ab}\gamma_{ab}\epsilon^i-V_\mu^{ij}\epsilon_j-\mathrm{i}\gamma_\mu\eta^i\\
&\quad +\mathrm{i}\gamma^{ab}T_{ab}\gamma_\mu\epsilon^i\,,\\
\delta_{Q,S}(\epsilon,\eta)T_{ab} &= \tfrac12\mathrm{i}\bar\epsilon\gamma_{ab}\chi-\tfrac{3}{32}\mathrm{i}\bar\epsilon\widehat R_{ab}(Q)\,,\\
\delta_{Q,S}(\epsilon,\eta)\chi^i &= \tfrac14\epsilon^i D-\tfrac{1}{64}\gamma^{ab}\widehat R_{ab}^{ij}(V)\epsilon_j+\tfrac18\mathrm{i}\gamma^{ab}\not{D}T_{ab}\epsilon^i-\tfrac18\mathrm{i}\gamma^a D^b T_{ab}\epsilon^i-\\
&\quad -\tfrac14\gamma^{abcd}T_{ab}T_{cd}\epsilon^i+\tfrac16 T^2\epsilon^i+\tfrac14\gamma^{ab}T_{ab}\eta^i\,,\\
\delta_{Q,S}(\epsilon,\eta)D &= \bar\epsilon\not{D}\chi-\tfrac53\mathrm{i}\bar\epsilon\gamma^{ab}T_{ab}\chi-\mathrm{i}\bar\eta\chi\,.
\end{aligned}
\tag{3.75}
$$

In 5 dimensions, not much simplifications are possible by taking appropriate C in (3.51) or ρ in (3.52), so that we just took the constraints [29]

$$
R^a_{\mu\nu}(P)=0\,,\qquad \gamma^b\widehat R_{ba}(Q)^i=0\,,\qquad \widehat R_{ac}(M^{bc})=0\,.
\tag{3.76}
$$

The full commutator of two supersymmetry transformations is

$$
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]=\delta_{\rm cgct}(\xi_3^\mu)+\delta_M(\lambda_3^{ab})+\delta_S(\eta_3)+\delta_U(\lambda_3^{ij})+\delta_K(\Lambda^a_{K3})\,.
\tag{3.77}
$$

The covariant general coordinate transformations have been defined in (3.2). The parameters appearing in (3.77) are

$$
\begin{aligned}
\xi_3^\mu &= \tfrac12\bar\epsilon_2\gamma_\mu\epsilon_1\,,\\
\lambda_3^{ab} &= -\mathrm{i}\bar\epsilon_2\gamma^{[a}\gamma^{cd}T_{cd}\gamma^{b]}\epsilon_1\,,\\
\lambda_3^{ij} &= \mathrm{i}\bar\epsilon_2^{(i}\gamma^{ab}T_{ab}\epsilon_1^{j)}\,,\\
\eta_3^i &= -\tfrac94\mathrm{i}\,\bar\epsilon_2\epsilon_1\chi^i+\tfrac74\mathrm{i}\,\bar\epsilon_2\gamma_c\epsilon_1\gamma^c\chi^i+\\
&\quad +\tfrac14\mathrm{i}\,\bar\epsilon_2^{(i}\gamma_{cd}\epsilon_1^{j)}\left(\gamma^{cd}\chi_j+\tfrac14\,\widehat R^{cd}{}_j(Q)\right)\,,\\
\Lambda^a_{K3} &= -\tfrac12\bar\epsilon_2\gamma^a\epsilon_1 D+\tfrac{1}{96}\bar\epsilon_2^i\gamma^{abc}\epsilon_1^j\widehat R_{bcij}(V)+\\
&\quad +\tfrac{1}{12}\mathrm{i}\bar\epsilon_2\left(-5\gamma^{abcd}D_bT_{cd}+9D_bT^{ba}\right)\epsilon_1+\\
&\quad +\bar\epsilon_2\left(\gamma^{abcde}T_{bc}T_{de}-4\gamma^cT_{cd}T^{ad}+\tfrac23\gamma^aT^2\right)\epsilon_1\,.
\end{aligned}
\tag{3.78}
$$

[12] For $\delta\chi$ the split in two lines is accidental due to the length of the expression.

For the Q, S commutators we find the following algebra:

$$\begin{array}{rcl} [\delta_S(\eta), \delta_Q(\epsilon)] &=& \delta_D(\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\eta) + \delta_M(\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma^{ab}\eta) + \delta_U(-\tfrac{3}{2}\mathrm{i}\bar{\epsilon}^{(i}\eta^{j)}) + \delta_K(\Lambda^a_{3K})\,, \\ [\delta_S(\eta_1), \delta_S(\eta_2)] &=& \delta_K(\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1)\,, \end{array} \qquad (3.79)$$

with

$$\Lambda^a_{3K} = \tfrac{1}{6}\bar{\epsilon}\left(\gamma^{bc}T_{bc}\gamma_a - \tfrac{1}{2}\gamma_a\gamma^{bc}T_{bc}\right)\eta\,. \qquad (3.80)$$

For practical purposes (see how to calculate transformations of covariant derivatives), it is useful to give the extra parts of the transformation laws of dependent gauge fields, i.e. the parts denoted by ${M_{\mu J}}^I$ in (3.8). These are (for the gauge field of special conformal transformations we suffice by giving the transformation of the trace, as this is what one often needs):

$$\begin{array}{rcl} \delta_{Q,S}(\epsilon,\eta){\omega_\mu}^{ab} &=& \cdots - \tfrac{1}{2}\bar{\epsilon}\gamma^{[a}{\widehat{R}_\mu}{}^{b]}(Q) - \tfrac{1}{4}\bar{\epsilon}\gamma_\mu\widehat{R}^{ab}(Q) - 4{e_\mu}^{[a}\bar{\epsilon}\gamma^{b]}\chi\,, \\ \delta_{Q,S}(\epsilon,\eta)\phi^i_\mu &=& \cdots - \tfrac{1}{12}\mathrm{i}\left\{\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab}\right\}{\widehat{R}_{ab}}{}^i{}_j(V)\epsilon^j + \\ && +\tfrac{1}{3}\left[\not{D}\gamma^{ab}T_{ab}\gamma_\mu - D_\mu\gamma^{ab}T_{ab} + \gamma_\mu\gamma^c D^a T_{ac}\right]\epsilon^i + \\ && +\mathrm{i}\left[-\gamma_\mu(\gamma^{ab}T_{ab})^2 + 4\gamma_c {T_\mu}^c\gamma^{ab}T_{ab} + 16\gamma_c T^{cd}T_{\mu d} - 4\gamma_\mu T^2\right]\epsilon^i + \\ && +\tfrac{1}{3}\mathrm{i}\left(8\gamma^b T_{\mu b} - \gamma_\mu\gamma\cdot T\right)\eta^i\,, \\ \delta_S(\eta){f_a}^a &=& -5\mathrm{i}\bar{\eta}\chi, \end{array} \qquad (3.81)$$

with $\gamma\cdot T = \gamma^{ab}T_{ab}$ and $T^2 = T^{ab}T_{ab}$. Note that there are other terms determined by the algebra. E.g. the λ_3^{ab} expression in (3.78) implies that the supersymmetry transformation of the spin connection contains a term

$$\delta_Q(\epsilon){\omega_\mu}^{ab} = \cdots - \mathrm{i}\bar{\epsilon}\gamma^{[a}\gamma^{cd}T_{cd}\gamma^{b]}\psi_\mu\,. \qquad (3.82)$$

3.5.4 $d = 6$

The transformation laws of the independent fields are

$$\begin{array}{rcl} \delta_{Q,S}(\epsilon,\eta){e_\mu}^a &=& \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\ \delta_{Q,S}(\epsilon,\eta)b_\mu &=& -\tfrac{1}{2}\bar{\epsilon}\phi_\mu + \tfrac{1}{2}\bar{\eta}\psi_\mu \\ && -\tfrac{1}{24}\bar{\epsilon}\gamma_\mu\chi\,, \\ \delta_{Q,S}(\epsilon,\eta){\mathcal{V}_\mu}^{ij} &=& 2\bar{\epsilon}^{(i}\phi^{j)}_\mu + 2\bar{\eta}^{(i}\psi^{j)}_\mu \\ && +\tfrac{1}{6}\bar{\epsilon}^{(i}\gamma_\mu\chi^{j)}\,, \\ \delta_{Q,S}(\epsilon,\eta)\psi^i_\mu &=& \left(\partial_\mu + \tfrac{1}{2}b_\mu + \tfrac{1}{4}\gamma^{ab}\omega_{\mu ab}\right)\epsilon^i + {V_\mu}^i{}_j\epsilon^j \\ && +\tfrac{1}{24}\gamma\cdot T^-\gamma_\mu\epsilon^i + \gamma_\mu\eta^i\,, \\ \delta_{Q,S}(\epsilon,\eta)T^-_{abc} &=& -\tfrac{1}{32}\bar{\epsilon}\gamma^{de}\gamma_{abc}\widehat{R}_{de}(Q) - \tfrac{7}{96}\bar{\epsilon}\gamma_{abc}\chi\,, \\ \delta_{Q,S}(\epsilon,\eta)\chi^i &=& +\tfrac{1}{8}(D_\mu\gamma\cdot T^-)\gamma^\mu\epsilon^i - \tfrac{3}{8}\widehat{R}^{ij}(V)\cdot\gamma\epsilon_j + \tfrac{1}{4}D\,\epsilon^i + \tfrac{1}{2}\gamma\cdot T^-\eta^i\,, \\ \delta_{Q,S}(\epsilon,\eta)D &=& \bar{\epsilon}^i\not{D}\chi_i - 2\bar{\eta}\chi\,, \end{array} \qquad (3.83)$$

where $\gamma\cdot T\equiv\gamma^{abc}T_{abc},\ldots$.

The constraints that we took in 6 dimensions are:

$$\begin{aligned} 0 &= R^a_{\mu\nu}(P)\,,\\ 0 &= \gamma^b\widehat{R}_{ba}(Q)^i+\tfrac16\gamma_a\chi^i\,,\\ 0 &= \widehat{R}_{ac}(M^{bc})-T^-_{acd}T^{-bcd}+\tfrac1{12}\delta_a{}^bD\,. \end{aligned}\tag{3.84}$$

The last equation contains a sign correction[13] to the equation in [30].

The transformations of the dependent gauge fields $\omega_\mu{}^{ab}$ and ϕ^i_μ contain as covariant terms (terms not determined by the algebra)

$$\begin{aligned} \delta_Q(\epsilon)\omega_\mu{}^{ab} &= \cdots-\tfrac12\bar\epsilon\gamma^{[a}\widehat{R}_\mu{}^{b]}(Q)-\tfrac14\bar\epsilon\gamma_\mu\widehat{R}^{ab}(Q)-\tfrac1{12}e_\mu{}^{[a}\bar\epsilon\gamma^{b]}\chi\,,\\ \delta_Q(\epsilon)\phi^i_\mu &= \cdots+\tfrac1{32}\left\{\gamma^{ab}\gamma_\mu-\tfrac12\gamma_\mu\gamma^{ab}\right\}\widehat{R}_{ab}{}^i{}_j(V)\epsilon^j+\\ &\quad -\tfrac1{96}\left(\not{D}\gamma^{abc}T^-_{abc}\gamma_\mu\right)\epsilon^i\,. \end{aligned}\tag{3.85}$$

In this case there are no extra S-supersymmetry transformations.

The algebra is only modified in the anticommutator of two Q-supersymmetries:

$$\begin{aligned} [\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)] &= \delta_{\rm cgct}\left(\tfrac12\bar\epsilon_2\gamma_\mu\epsilon_1\right)+\delta_M\left(\tfrac12\bar\epsilon_2\gamma_c\epsilon_1T^{-\,abc}\right)\\ &\quad +\delta_S\left(\tfrac1{24}\gamma_a\chi^i\bar\epsilon_2\gamma^a\epsilon_1\right)+\delta_K\left(-\tfrac18\bar\epsilon_2\gamma_b\epsilon_1\left(D_cT^{-\,abc}+\tfrac1{12}\eta^{ab}D\right)\right)\,. \end{aligned}\tag{3.86}$$

Again, this implies that the transformations of gauge fields contain extra terms as e.g. there is in the transformation of the spin connection $+\frac12\bar\epsilon\gamma_c\psi_\mu T^{-\,abc}$.

4 Matter multiplets

Our general strategy has been explained in section 2.3. In a first subsection we will expand on this using the better knowledge that we have now on the conformal group and its gauge fields. Then we can introduce various matter multiplets. First we explain their transformations under all the symmetries, and then we discuss actions.

4.1 Example in bosonic case

In section 2.3, we already outlined the general idea of the superconformal construction for actions with only super-Poincaré invariance. At that time, we had not yet explained the gauging of the conformal algebra. Now we can be more precise. For this example, we will still restrict to the bosonic case. Consider a scalar field ϕ with Weyl weight w. Its superconformal covariant derivative is thus

$$D_\mu\phi=\left(\partial_\mu-wb_\mu\right)\phi\,.\tag{4.1}$$

[13] I thank T. Kugo for this correction.

The transformation of the covariant derivative $D_a\phi$ can be easily obtained from the simple method. One takes into account (3.10) to find that there is a K transformation. The transformation law of a covariant derivative determines the covariant box

$$\begin{aligned}\Box^C\phi &\equiv \eta^{ab}D_bD_a\phi = e^{a\mu}\left(\partial_\mu D_a\phi - (w+1)b_\mu D_a\phi + \omega_{\mu\,ab}D^b\phi + 2wf_{\mu a}\phi\right)\\ &= e^{-1}\left(\partial_\mu - (w+2-d)b_\mu\right)eg^{\mu\nu}\left(\partial_\nu - wb_\nu\right)\phi - \frac{w}{2(d-1)}R\,\phi.\end{aligned} \tag{4.2}$$

I use here the constraint (3.55) (without matter for the pure bosonic case). The last term is the well-known $\mathcal{R}/6$ term in $d=4$. In fact, choosing $w=\frac{d}{2}-1$, one has a conformal invariant scalar action

$$S = \int \mathrm{d}^dx\, e\phi\Box^C\phi\,. \tag{4.3}$$

Exercise 4.1: Show that $\int \mathrm{d}^dx\, eD_a\phi\, D^a\phi$ is not a special conformal invariant. However, $\Box\phi$ is invariant under K iff $w=\frac{d}{2}-1$.

Consider now that we want a Poincaré invariant action. Then we have to break dilatations and special conformal transformations, as these are not part of the Poincaré algebra. Considering (3.10), it is clear that the latter can be broken by a gauge choice

$$K-\text{gauge}:\qquad b_\mu = 0\,. \tag{4.4}$$

Therefore, of the 'Weyl multiplet' (the multiplet of fields with the gauge fields of the conformal algebra), only the vielbein remains. One could take as gauge choice for dilatations a fixed value of a scalar ϕ. One easily checks that then the action (4.3) reduces to the Poincaré gravity action.

I can schematically summarize this procedure in the following diagram:

$$\begin{array}{cl}\text{Weyl multiplet: } e_\mu{}^a,\ b_\mu & \\ + & \\ \text{matter field: } \phi & \\ \downarrow & \text{gauge fixing } K_a\,D \\ \text{Poincaré gravity } e_\mu{}^a\,. & \end{array} \tag{4.5}$$

The field ϕ is called a compensating field.

XXX Stress that I do not construct the conformal action for gravity

XXX Explain that gauge fixing is equivalent to redefinitions.

4.2 Conformal properties of the multiplets.

We have defined the Weyl multiplet, and the algebra, which depends for part on the fields of the Weyl multiplet, is fixed for what concerns the superconformal transformations. We will see below that extra terms with gauge transformations of extra vectors or antisymmetric tensors may still appear. The fields of the Weyl multiplet are inert under these transformations, and these extra transformations do therefore not modify the previous results.

All fields in 'matter multiplets' will now have to obey the same algebra. A first step is to define their transformations under the bosonic symmetries. General coordinate transformations XXX Lorentz rotations XXX Weyl transformations. Usually,

$$\delta_D(\Lambda_D)\phi = w\Lambda_D\phi\,. \tag{4.6}$$

R-symmetry, see indices for SU(2), and for U(1) (complex XXX)

$$\delta_A(\Lambda_A)\phi = \mathrm{i}c\Lambda_A\phi\,. \tag{4.7}$$

The number w is called the Weyl weight and c the chiral weight.

In the previous chapter, the Weyl multiplet is constructed, which is the background for the matter multiplets. Other multiplets should now transform in the representation of the 'soft group' defined by the Weyl multiplet. As I have already mentioned, a gauge vector multiplet may modify this structure. The commutator of the supersymmetries can still be modified by a gauge transformation that depends on fields of this vector multiplet. For this structure to make sense, the algebra of the Weyl multiplet had to close without using an equation of motion. A matter multiplet may now be introduced whose algebra closes only modulo equations of motion. However, consider now the situation that there is matter transforming in a representation of a Yang–Mills group. Then, in order to define the matter multiplet in the background of the Yang–Mills multiplet, the latter has to be well defined off-shell. Therefore we will have to start with the vector multiplet.

4.2.1 Vector multiplets

The vector multiplet for $d = 6$ has already been introduced in section 3.2.2. The transformations are given in (3.24). Later, it was shown in (3.41) that the supersymmetry transformations do not close. The solution is well-known. The 5 bosonic components of the gauge vector, and the 8 components of the spinor, need an SU(2)-triplet of real scalars, $Y^{(ij)}$. This will appear in the transformation law of the fermion. As an example, I will determine here its transformation law, using principles that can be used in general.

In general, it is useful to consider first the Weyl weights of the fields. I will give here also the argument for the weight $\frac{3}{2}$ of the gaugino. The starting point is the vector. It is a gauge vector that gauges a U(1) group, *commuting* with the supersymmetry algebra. The superconformal group is complete as it is given, and all other transformations have to commute with it. Indeed, all extra transformations beyond those in the superconformal algebra commute with the latter[14]. This implies that formally the parameters have to be considered as Weyl weight 0. (In principle parameters do not transform, but the commutators of symmetries can be stated in these terms). Consider then the transformation $\delta W_\mu = \partial_\mu\alpha$.

[14]There may still be a field-dependent correction of commutators of the superconformal algebra that involve other transformations, that are similar to central charges as we will see below.

Table 4: *Fields in some superconformal matter multiplets. I indicate for each dimension the Weyl weight (and for d = 4 chiral weight), the number of real degrees of freedom, the* SU(2) *representations, which is the same in any dimension, and the chirality for d = 4 and d = 6. For each multiplet I give first the bosonic fields, and then the fermionic fields (below the line).*

d = 4				**d = 5**			**d = 6**				
field	w	c	#	field	w	#	field	w	#	SU(2)	γ_5/γ_7
Off-shell vector multiplet											
X	1	-1	2	σ	1	1				1	
W_μ	0	0	3	A_μ	0	4	W_μ	0	5	1	
Y_{ij}	2	0	3	Y_{ij}	3	3	Y_{ij}	3	3	3	
Ω_i	3/2	$-1/2$	8	ψ_i	3/2	8	Ω_i	3/2	8	2	+
				On-shell tensor multiplet							
				$B_{\mu\nu}$	0	3	$B_{\mu\nu}$	0	3	1	
				ϕ	1	1	σ	2	1	1	
				λ^i	3/2	4	ψ^i	5/2	4	2	
On-shell hypermultiplet											
q^X	1	0	4	q^X	3/2	4	q^X	2	4	2	
ζ^A	3/2	$-1/2$	4	ζ^A	2	4	ζ^A	5/2	4	1	+
Off-shell chiral multiplet											
A	w	$-w$	2							1	
B_{ij}	$w+1$	$-w+1$	6							3	
G^-_{ab}	$w+1$	$-w+1$	6							1	
C	$w+2$	$-w+2$	2							1	
Ψ_i	$w+\frac{1}{2}$	$-w+\frac{1}{2}$	8							2	+
Λ_i	$w+\frac{3}{2}$	$-w+\frac{3}{2}$	8							2	$-$
Off-shell linear multiplet											
L_{ij}	2	0	3	L_{ij}	3	3	L_{ij}	4	3	3	
E_a	3	0	3	E_a	4	4	E_a	5	5	1	
G	3	1	2	N	4	1				1	
φ_i	5/2	$-1/2$	8	φ^i	7/2	8	φ^i	9/2	8	2	$-$

This implies that W_μ has Weyl weight 0. The same argument holds in fact for any gauge field, or gauge two-form, ... The the curvature F_{ab} has Weyl weight 2, due to the vielbeins involved in $F_{ab} = e_a{}^\mu e_b{}^\nu F_{\mu\nu}$. As I have explained, these are the covariant quantities that should appear in the transformations of other matter fields. the supersymmetry parameter ϵ should be considered to be of Weyl weight $-\frac{1}{2}$, as its gauge field ψ_μ. Thus the supersymmetry transformation of the gaugino to the field strength of the gauge field determines that the conformal weight of Ω is indeed $\frac{3}{2}$.

Exercise 4.2: Determine the same result from the transformation of the gauge field to the gaugino.

The only way in which we can involve the auxiliary field Y^{ij} in the transformation is with an extra term $\delta\Omega^i = Y^{ij}\epsilon_j$. The auxiliary field should then be of Weyl weight 2. In its supersymmetry transformation law can appear a covariant fermionic object of Weyl weight $\frac{5}{2}$. This is consistent with a transformation to the covariant derivative of the gaugino, in order to cancel (3.41). The full transformation laws are

$$\begin{aligned}
\delta W_\mu &= \partial_\mu\alpha - \bar\epsilon\gamma_\mu\Omega\,,\\
\delta\Omega^i &= \left(\tfrac{3}{2}\Lambda_D - \tfrac{1}{4}\gamma^{ab}\lambda_{ab}\right)\Omega^i + \Lambda^{ij}\Omega_j + \tfrac{1}{8}\gamma^{ab}\widehat{F}_{ab}\epsilon^i - \tfrac{1}{2}Y^{ij}\epsilon_j\\
\delta Y^{ij} &= 2\Lambda_D Y^{ij} + 2\Lambda^{k(i}Y^{j)}{}_k - \bar\epsilon^{(i}\not{D}\Omega^{j)} + 2\bar\eta^{(i}\Omega^{j)}\,.
\end{aligned}\tag{4.8}$$

The final term is the only S-transformation that can occur consistent with Weyl weights. Its coefficient has to be fixed from calculating the $[\delta_Q(\epsilon),\delta_Q(\eta)]$ commutator on the gaugino. One can check that the extra terms from Y^{ij} cancel the non-closure terms (3.41).

Exercise 4.3: Check that all the transformation laws determine (and are consistent with) Ω to be a left-chiral spinor, in accordance with table 1.

Consider now the *vector multiplet in 4 dimensions*. It could be obtained from dimensional reduction of this one in 6 dimensions. As mentioned already in section 2.2, it has then a complex scalar. These are the fourth and fifth components of the vector of 6 dimensions. To get the right behaviour of gauge and general coordinate transformations, one has to consider the vector with tangent spacetime indices to consider the reduction (see [12, 13] and the lectures of C. Pope XXX in this school). $e_a{}^\mu W_\mu$ has Weyl weight 1, and this is therefore the Weyl weight of the complex scalar X that appears here.

There is even more. Remember that the covariant general coordinate transformations contain a linear combination of all gauge symmetries. That involves also the gauge transformation of the vector. Thus in the commutator of two supersymmetry transformations is a term $\bar\epsilon_2\gamma^\mu\epsilon_1 W_\mu$. When reduced to 4 dimensions, some components of W_μ are replaced by the scalars X. This is the origin of a new term in the supersymmetry commutator involving structure functions depending on the scalars. Before giving the supersymmetry transformations, I have to translate the reality of the triplet Y_{ij} in appropriate notation for 4 dimensions. In 6 dimensions the reality is $Y = Y^* = \sigma_2 Y^C\sigma_2$. It is in the form with Y^C that we have to translate it, thus giving rise to

$$Y_{ij} = \varepsilon_{ik}\varepsilon_{j\ell}Y^{k\ell}\,,\qquad Y^{ij} = (Y_{ij})^*\,.\tag{4.9}$$

I will introduce here at once the vector multiplet in a non-Abelian group.[15] Note that I will use the index I from now on to enumerate the vector multiplets, and thus the generators of the non-Abelian algebra that can be gauged. I hope that this does not lead to confusion with the index I that was used so far to denote all the generators including supersymmetry, ... as it was done in section 3.

The transformations under dilatations and chiral U(1) transformations follow from table 4, with the general rules (4.6) and (4.7). The supersymmetry (Q and S), and the gauge transformations with parameter α of the multiplet in 4 dimensions are

$$\begin{aligned}
\delta X^I &= \tfrac{1}{2}\bar{\epsilon}^i\Omega_i^I - g\alpha^J X^K f_{JK}{}^I, \\
\delta\Omega_i^I &= \not{D}X^I\epsilon_i + \tfrac{1}{4}\gamma^{ab}\mathcal{F}_{ab}^{I-}\varepsilon_{ij}\epsilon^j + \tfrac{1}{2}Y_{ij}^I\epsilon^j + gX^J\bar{X}^K f_{JK}{}^I\varepsilon_{ij}\epsilon^j \\
&\quad +2X^I\eta_i - g\alpha^J\Omega_i^K f_{JK}{}^I, \\
\delta W_\mu^I &= \tfrac{1}{2}\varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j^I + \varepsilon^{ij}\bar{\epsilon}_i\psi_{\mu j}X^I + \text{h.c.} + \partial_\mu\alpha^I - g\alpha^J W_\mu^K f_{JK}{}^I, \\
\delta Y_{ij}^I &= \bar{\epsilon}_{(i}\not{D}\Omega_{j)}^I + \varepsilon_{ik}\varepsilon_{j\ell}\bar{\epsilon}^{(k}\not{D}\Omega^{\ell)I} + 2g\varepsilon_{k(i}\left(\bar{\epsilon}_{j)}X^J\Omega^{kK} - \bar{\epsilon}^k\bar{X}^J\Omega_{j)K}\right)f_{JK}{}^I \\
&\quad -g\alpha^I Y_{ij}^K f_{JK}{}^I,
\end{aligned} \tag{4.10}$$

where

$$\mathcal{F}_{ab}^{I-} \equiv \widehat{F}_{ab}^{I-} - \tfrac{1}{2}\bar{X}^I T_{ab}^-. \tag{4.11}$$

In the latter expression, the $\widehat{F}$ is covariant with the new structure functions, as dictated by definitions given in section 3. Indeed, the second term of the transformation of the vector reflects the presence of the new term in the commutator of two supersymmetries, modifying (3.73)

$$\begin{aligned}
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_P\left(\xi^a(\epsilon_1,\epsilon_2)\right) + \delta_M\left(\lambda^{ab}(\epsilon_1,\epsilon_2)\right) + \delta_K\left(\Lambda_K^a(\epsilon_1,\epsilon_2)\right) + \delta_S\left(\eta(\epsilon_1,\epsilon_2)\right) \\
&\quad +\delta_G\left(\alpha^I(\epsilon_1,\epsilon_2) = \varepsilon^{ij}\bar{\epsilon}_{2i}\epsilon_{1j}X^I + \text{h.c.}\right),
\end{aligned} \tag{4.12}$$

where δ_G is the gauge transformation that I announced before.

Exercise 4.4: Check that this leads to the form of $\widehat{F}_{\mu\nu}{}^I$ as given below in (4.83).

For the tensor calculus that is used to construct actions for the vector multiplet, it is important to note that a vector multiplet is a constrained chiral multiplet.

4.2.2 Chiral multiplet

A multiplet corresponds to a superfield. It can be constrained or unconstrained. A superfield can be real or chiral, or carry a Lorentz representation, or be in a non-trivial representations of the SU(2) automorphism group, ... A chiral superfield is e.g. a superfield that is a Lorentz scalar, complex, and satisfies the constraint that one chiral superspace derivative is zero.

[15] To be in accordance with common practice here, I denote the complex conjugates of the scalar fields by $\bar{X}$ rather than X^*.

All that can be directly translated in components. A multiplet is defined by its 'lowest' component. In superconformal language, 'lowest' is with respect to the Weyl weight. Without the superconformal group, one may refer to engineering dimensions. That lowest component may be a scalar, or any other representation. For a chiral multiplet it is a complex scalar. An unconstrained multiplet would mean that I allow the lowest component to transform in arbitrary spinors. Then the transformations of these arbitrary spinors can have arbitrary expressions containing new fields, as long as it is consistent with the algebra. See e.g. section 3.1 of [46] for the example of $N = 1$ chiral multiplets, and section 2.2 of [47] for $N = 2$.

A chiral superfield is then the restriction that the lowest (complex) component (A) transforms with only the left supersymmetry:

$$\delta_Q(\epsilon)A = \tfrac{1}{2}\bar\epsilon^i\Psi_i\,. \tag{4.13}$$

Thus the transformation does not contain ϵ_i, the right-chiral component. Imposing the rigid supersymmetry algebra leads to the following general expressions:

$$\begin{aligned}
\delta_Q(\epsilon)A &= \tfrac{1}{2}\bar\epsilon^i\Psi_i\,,\\
\delta_Q(\epsilon)\Psi_i &= \partial\!\!\!/ A\epsilon_i + \tfrac{1}{2}B_{ij}\epsilon^j + \tfrac{1}{4}\gamma_{ab}G^{-ab}\varepsilon_{ij}\epsilon^j\,,\\
\delta_Q(\epsilon)B_{ij} &= \bar\epsilon_{(i}\partial\!\!\!/\Psi_{j)} - \bar\epsilon^k\Lambda_{(i}\varepsilon_{j)k}\,,\\
\delta_Q(\epsilon)G^-_{ab} &= \tfrac{1}{4}\varepsilon^{ij}\bar\epsilon_i\partial\!\!\!/\gamma_{ab}\Psi_i + \tfrac{1}{4}\bar\epsilon^i\gamma_{ab}\Lambda_i\,,\\
\delta_Q(\epsilon)\Lambda_i &= -\tfrac{1}{4}\gamma^{ab}G^-_{ab}\overleftarrow{\partial\!\!\!/}\,\epsilon_i - \tfrac{1}{2}\partial\!\!\!/ B_{ij}\varepsilon^{jk}\epsilon_k + \tfrac{1}{2}C\varepsilon_{ij}\epsilon^j\,,\\
\delta_Q(\epsilon)C &= -\varepsilon^{ij}\bar\epsilon_i\partial\!\!\!/\Lambda_j\,.
\end{aligned} \tag{4.14}$$

You can count that this is a 16 + 16 multiplet counted as real components. In fact, it is reducible. This can be seen by that one can impose the following conditions:

$$\begin{aligned}
&B_{ij} - \varepsilon_{ik}\varepsilon_{j\ell}B^{k\ell} = 0\,,\\
&\partial\!\!\!/\Psi^i - \varepsilon^{ij}\Lambda_j = 0\,,\\
&\partial_b(G^{+ab} - G^{-ab}) = 0\,,\\
&C - 2\partial_a\partial^a\bar A = 0\,,
\end{aligned} \tag{4.15}$$

where $B^{k\ell}$ is, as usual, defined by the complex conjugate of $B_{k\ell}$, and similarly G^+ is the complex conjugate of G^-, and thus self-dual as G^- is antiselfdual. These constraints are consistent in the sense that a supersymmetry variation of one of them leads to the other equations, and this is a complete set in that sense. The third equation is a Bianchi identity that can be solved by interpreting G_{ab} as the field strength of a vector. The independent components are then those of the vector multiplet, identifying

$$X = A\,,\qquad \Omega_i = \Psi_i\,,\qquad F_{ab} = G_{ab}\,,\qquad Y_{ij} = B_{ij}\,. \tag{4.16}$$

You can identify the linear part of (4.10) with the transformations of the vector multiplet. Now, I want to define all this in the full superconformal algebra.

To define the chiral multiplet in the conformal algebra, one first allows an arbitrary Weyl weight for A, say that this is w. Then you may allow a general S-supersymmetry transformation for Ψ_i, and consistency with the Weyl weights imposes that this should be proportional to A. Imposing the $\{Q,S\}$ algebra immediately shows that the chiral U(1) weight of A should be related to its Weyl weight. In fact, to avoid the ϵ_i terms in this anticommutator, one should define that under dilatations and U(1),

$$\delta_{D,U(1)}(\Lambda_D,\Lambda_A)A = w\,(\Lambda_D - \mathrm{i}\Lambda_A)\,A\,. \tag{4.17}$$

You may then determine the same transformations for the other fields from compatibility with the supersymmetries, and you obtain

$$\begin{aligned}
\delta_{D,U(1)}(\Lambda_D,\Lambda_A)\Psi_i &= \left((w+\tfrac12)\Lambda_D + \mathrm{i}(-w+\tfrac12)\Lambda_A\right)\Psi_i\,,\\
\delta_{D,U(1)}(\Lambda_D,\Lambda_A)B_{ij} &= \left((w+1)\Lambda_D + \mathrm{i}(-w+1)\Lambda_A\right)B_{ij}\,,\\
\delta_{D,U(1)}(\Lambda_D,\Lambda_A)G^-_{ab} &= \left((w+1)\Lambda_D + \mathrm{i}(-w+1)\Lambda_A\right)G^-_{ab}\,,\\
\delta_{D,U(1)}(\Lambda_D,\Lambda_A)\Lambda_i &= \left((w+\tfrac32)\Lambda_D + \mathrm{i}(-w+\tfrac32)\Lambda_A\right)\Lambda_i\,,\\
\delta_{D,U(1)}(\Lambda_D,\Lambda_A)C &= \left((w+2)\Lambda_D + \mathrm{i}(-w+2)\Lambda_A\right)C\,.
\end{aligned}\tag{4.18}$$

To complete the superconformal multiplet, one has to add S-transformations, and there are non-linear transformations involving the matter fields of the Weyl multiplet χ_i and T_{ab}, necessary in order to represent the anticommutators (3.73). The full result was found in [48]:

$$\begin{aligned}
\delta_{Q,S}(\epsilon,\eta)A &= \tfrac12\bar\epsilon^i\Psi_i\,,\\
\delta_{Q,S}(\epsilon,\eta)\Psi_i &= \not{D}A\epsilon_i + \tfrac12 B_{ij}\epsilon^j + \tfrac14\gamma\cdot G^-\varepsilon_{ij}\epsilon^j + 2wA\eta_i\,,\\
\delta_{Q,S}(\epsilon,\eta)B_{ij} &= \bar\epsilon_{(i}\not{D}\Psi_{j)} - \bar\epsilon^k\Lambda_{(i}\varepsilon_{j)k} + 2(1-w)\bar\eta_{(i}\Psi_{j)}\,,\\
\delta_{Q,S}(\epsilon,\eta)G^-_{ab} &= \tfrac14\varepsilon^{ij}\bar\epsilon_i\not{D}\gamma_{ab}\Psi_j + \tfrac14\bar\epsilon^i\gamma_{ab}\Lambda_i - \tfrac12\varepsilon^{ij}(1+w)\bar\eta_i\gamma_{ab}\Psi_j\,,\\
\delta_{Q,S}(\epsilon,\eta)\Lambda_i &= -\tfrac14\gamma\cdot G^-\overleftarrow{\not{D}}\epsilon_i - \tfrac12\not{D}B_{ij}\epsilon_k\varepsilon^{jk} + \tfrac12 C\epsilon^j\varepsilon_{ij}\\
&\quad -\tfrac18(\not{D}A)T\cdot\gamma)\epsilon_i - \tfrac18 wA(\not{D}T)\cdot\gamma\epsilon_i - \tfrac34(\bar\chi_{[i}\gamma_a\Psi_{j]})\gamma^a\epsilon_k\varepsilon^{jk}\\
&\quad -(1+w)B_{ij}\varepsilon^{jk}\eta_k + \tfrac12(1-w)\gamma\cdot G^-\eta_i\,,\\
\delta_{Q,S}(\epsilon,\eta)C &= -\varepsilon^{ij}\bar\epsilon_i\not{D}\Lambda_j - 3\bar\epsilon_i\chi_j B_{k\ell}\varepsilon^{ik}\varepsilon^{j\ell}\\
&\quad +\tfrac18(w-1)\bar\epsilon_i\gamma\cdot T\overleftarrow{\not{D}}\Psi_j\varepsilon^{ij} + \tfrac18\bar\epsilon_i\gamma\cdot T\not{D}\Psi_j\varepsilon^{ij} + 2w\varepsilon^{ij}\bar\eta_i\Lambda_j\,.
\end{aligned}\tag{4.19}$$

The constraints (4.15) are, however, not consistent with any choice of w. E.g. the first constraint, is a reality condition, and it is easy to check that this is only consistent if the chiral weight of B_{ij} is zero. This fixes $w=1$. That is the appropriate value also to interpret G_{ab} as a covariant field strength. The full constraints are

$$\begin{aligned}
0 &= B_{ij} - \varepsilon_{ik}\varepsilon_{jl}B^{kl}\,,\\
0 &= \not{D}\Psi^i - \varepsilon^{ij}\Lambda_j\,,\\
0 &= D^a(G^+_{ab} - G^-_{ab} + \tfrac12 AT_{ab} - \tfrac12\bar A T_{ab}) - \tfrac34(\varepsilon^{ij}\bar\chi_i\gamma_b\Psi_j - \text{h.c.})\,,\\
0 &= -2\Box\bar A - \tfrac12 G^+_{\mu\nu}T^{\mu\nu} - 3\bar\chi_i\Psi^i - C\,.
\end{aligned}\tag{4.20}$$

Note that the Bianchi identity shows the shift between the pure covariant field strength and the G. Compare this with (4.11).

The chiral multiplet is so useful because the upper component is a scalar. In the rigid supersymmetry case, it transforms to a total derivative. Therefore it can serve as an action. That action corresponds in superspace to taking the full chiral superspace integral of the chiral superfield. In order to do so in the superconformal context, one has to think of an action of the form

$$I = \int \mathrm{d}^4 x \, e \, C + \ldots + \text{h.c.} \,. \tag{4.21}$$

The Weyl weight of the determinant of the vielbein is -4, so C should have Weyl weight 4. It should also have chiral weight 0. Both these requirements lead to a chiral multiplet with $w = 2$. The $+\ldots$ in (4.21) are terms that should change the transformation of C that is covariant, i.e. $\bar{\epsilon}\not{D}\Lambda$ to a total derivative. So these terms contain explicit gauge fields. The full expression has been obtained in [48].

4.2.3 Hypermultiplets

Hypermultiplets are defined in the background of the Weyl multiplet and possibly also in the background of the vector multiplet. The latter is the case if one considers hypermultiplets that transform non-trivially under the gauge transformations of the vector multiplets. Auxiliary fields to close the algebra (in the sense explained before that 'open' means closed including the trivial symmetries) exist for the simplest quaternionic manifolds, or can be introduced if one uses the methods of harmonic superspace. However, we can avoid this. We do not need auxiliary fields any more at this point. This is because the hypermultiplets are at the end of the hierarchy line. We are not going to introduce any further multiplet in the background of the hypermultiplets, as these do not introduce new gauge symmetries. When we considered the vector multiplets, the construction had to take into account that the multiplets can be used for various possible actions (including hypermultiplets or not).

The algebra thus closes only modulo 'equations of motion'. The closure of the supersymmetry algebra will impose equations that we will interpret as equations of motion, even though we have not defined an action yet. Later we will see how an action can be constructed that gives precisely these equations as Euler-Lagrange equations.

Rigid transformations in 5 dimensions. We start with $4r$ scalar fields and $2r$ spinors. The scalars are denoted by q^X with $X = 1, \ldots, 4r$ and the spinors by ζ^A with $A = 1, \ldots, 2r$. We use the formulation in 5 dimensions, and first for rigid supersymmetry. For details on spinor properties and our conventions, we refer to [29]. We consider general transformations for the scalars under the two supersymmetries with parameters ϵ^i, $i = 1, 2$:

$$\delta_Q(\epsilon) q^X = -\mathrm{i}\bar{\epsilon}^i \zeta^A f_{iA}^X(q) \,, \tag{4.22}$$

where the 'vielbeins' $f_{iA}^X(q)$ satisfy a reality condition

$$\rho_{AB} E_{ij} (f_{jB}^X)^* = f_{iA}^X \,, \tag{4.23}$$

defined by matrices E_{ij} and ρ_{AB} that satisfy

$$E\,E^* = -\mathbb{1}_2\,, \qquad \rho\,\rho^* = -\mathbb{1}_{2r}\,. \tag{4.24}$$

One may choose a standard antisymmetric form for ρ and identify E with ε by a choice of basis. The transformations on variables with an A index are by the reality condition restricted to $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{SU}^*(2r)\times \mathrm{U}(1)$.

To realize the supersymmetry algebra (2.5) on the scalars we need the supersymmetry transformation of the fermions ζ^A. They should therefore be of the form

$$\delta_Q(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\partial\!\!\!/ q^X f^{iA}_X(q)\epsilon_i - \zeta^B \omega_{XB}{}^A(q)\left[\delta(\epsilon)q^X\right], \tag{4.25}$$

where the f that appear here have indices in opposite position as in (4.22), indicating that they are the inverse matrices as $4r\times 4r$ matrices

$$f^{iA}_Y f^X_{iA} = \delta_Y{}^X\,, \qquad f^{iA}_X f^X_{jB} = \delta_j{}^i\delta_B{}^A\,. \tag{4.26}$$

The last term in the fermion transformation is necessary to cancel the variation of the f coefficients in the commutator on q^X. This leads to an integrability condition:

$$\begin{aligned}\mathfrak{D}_Y f^X_{iA} &\equiv \partial_Y f^X_{iA} - \omega_{YA}{}^B(q) f^X_{iB} + \Gamma^X_{YZ}(q) f^Z_{iA} = 0,\\ \mathfrak{D}_Y f^{iA}_X &\equiv \partial_Y f^{iA}_X + f^{iB}_X\omega_{YB}{}^A(q) - \Gamma^Z_{YX}(q) f^{iA}_Z = 0,\end{aligned} \tag{4.27}$$

where $\Gamma^X_{ZY}(q) = \Gamma^X_{YZ}(q)$ is any symmetric function of the scalars. We will identify these equations as expressing a covariant constancy of the vielbeins.

This can then be interpreted geometrically. $\omega_{XB}{}^A(q)$ are seen as gauge fields for the $\mathrm{G}\ell(r,\mathbb{Q})$. Obviously, we can interpret $\Gamma^X_{YZ}(q)$ as an affine connection. Also complex structures can be defined as (using vector sign for the three complex structures and using the three sigma matrices)

$$\vec{J}_X{}^Y \equiv -\mathrm{i} f^{iA}_X \vec{\sigma}_i{}^j f^Y_{jA} \qquad \Rightarrow \qquad J_X{}^Y{}_i{}^j \equiv \mathrm{i}\vec{J}_X{}^Y\cdot\vec{\sigma}_i{}^j = 2f^{jA}_X f^Y_{iA} - \delta_i{}^j\delta_X{}^Y\,.\,. \tag{4.28}$$

The complex structures satisfy, due to (4.26), the quaternion algebra (using here for convenience indices x,y,z instead of the vector sign)

$$J^x J^y = -\mathbb{1}_{4r}\delta^{xy} + \varepsilon^{xyz}J^z\,. \tag{4.29}$$

This defines the space of the scalars to be a hypercomplex manifold. To get more insight in their structure, we consider the integrability equation of (4.27), i.e. acting on it with another derivative and antisymmetrizing. This equation involves the curvature of the manifold as defined by the connection $\Gamma_{XY}{}^Z$ and the $\mathrm{G}\ell(r,\mathbb{H})$ curvature

$$\begin{aligned}R_{XYZ}{}^W &\equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V\,,\\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C.\end{aligned} \tag{4.30}$$

The integrability conditions relate these two, and they are further determined by one tensor $W_{ABC}{}^D$, symmetric in the lower indices (for details on the derivations, see appendix B in [49])

$$\begin{aligned} R_{XYW}{}^Z &= f^Z_{iA} f^{iB}_W \mathcal{R}_{XYB}{}^A \\ \mathcal{R}_{XYB}{}^A &= -\tfrac{1}{2} f^{iC}_X f^{jD}_Y \varepsilon_{ij} W_{CDB}{}^A, \qquad R_{XYW}{}^Z = -\tfrac{1}{2} f^{Ai}_X \varepsilon_{ij} f^{jB}_Y f^{kC}_W f^Z_{kD} W_{ABC}{}^D. \end{aligned} \tag{4.31}$$

Exercise 4.5: Show that the previous formulae lead also to

$$\mathcal{R}_{XYB}{}^A f^X_{iC} f^Y_{jD} = -\tfrac{1}{2}\varepsilon_{ij} W_{BCD}{}^A \quad \text{or} \quad W_{BCD}{}^A \equiv \tfrac{1}{2}\varepsilon^{ij} f^X_{jB} f^Y_{iC} f^Z_{kD} f^{kA}_W R_{XYZ}{}^W. \tag{4.32}$$

The conclusion so far is that imposing the supersymmetry transformations on the bosons lead to the identification of a hypercomplex manifold parametrized by these bosons q^X. The structure is determined by vielbeins f^{iA}_X and connections $\omega_{XA}{}^B$ and Γ^Z_{XY} (the latter symmetric in its lower indices) such that (4.26) and (4.27) are satisfied. This leads to the identification of complex structures as in (4.28) and the curvature tensor (4.31).

Algebra on the fermions One can then check that with the given transformations and identities (4.26) and (4.27) the supersymmetry algebra on the fermions closes according to (2.5) modulo terms proportional to

$$\Gamma^A \equiv \not{\mathfrak{D}}\zeta^A + \frac{1}{2} W_{BCD}{}^A \zeta^B \bar{\zeta}^C \zeta^D, \qquad \mathfrak{D}_\mu \zeta^A \equiv \partial_\mu \zeta^A + (\partial_\mu q^X)\zeta^B \omega_{XB}{}^A. \tag{4.33}$$

There are more terms if the fields transform under the gauge group of a vector multiplet, see [49]. Putting this equal to zero (demanding an on-shell algebra) gives an equation of motion for the fermions. The supersymmetry transformation of this equation gives then also an equation of motion for the bosons. As announced before, we thus have already physical equations despite the absence of an action.

Intermezzo: Reparametrizations and covariant quantities. We could now wonder in how far the parametrization of the scalars and the fermions has been important. There are thus two kinds of reparametrizations. The first ones are the target space diffeomorphisms, $q^X \to \widetilde{q}^X(q)$, under which f^X_{iA} transforms as a vector, $\omega_{XA}{}^B$ as a one-form, and $\Gamma_{XY}{}^Z$ as a connection. The second set are the reparametrizations which act on the tangent space indices $A, B, \ldots$. On the fermions, they act as

$$\zeta^A \to \widetilde{\zeta}^A(q) = \zeta^B U_B{}^A(q)\,, \tag{4.34}$$

where $U_A{}^B(q)$ is an invertible matrix, and the reality conditions impose $U^* = \rho^{-1} U \rho$, defining $\mathrm{G}\ell(r,\mathbb{H})$. In general, such a transformation brings us into a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X

and ζ^A are independent fields. The supersymmetry transformation rules (4.22) and (4.25) are covariant under (4.34) if we transform $f_X^{iA}(q)$ as a vector and $\omega_{XA}{}^B$ as a connection,

$$\omega_{XA}{}^B \to \widetilde{\omega}_{XA}{}^B = [(\partial_X U^{-1})U + U^{-1}\omega_X U]_A{}^B\,. \tag{4.35}$$

These considerations lead us to define the covariant variation of vectors with indices in the tangent space, as ζ^A, or a quantity Δ^X with coordinate indices:

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B\omega_{XB}{}^A\delta q^X\,, \qquad \widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y\Gamma_{YZ}{}^X\delta q^Z\,, \tag{4.36}$$

for any transformation δ (as e.g. supersymmetry, conformal transformations,...).

Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (4.34) are equivalent; they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. The expression $\partial_X\zeta^A$ makes only sense if one compares different bases. But in the same way also the expression $\zeta^B\omega_{XB}{}^A$ makes only sense if one compares different bases, as the connection has no absolute value. The only covariant object is the covariant derivative

$$\mathfrak{D}_X\zeta^A \equiv \partial_X\zeta^A + \zeta^B\omega_{XB}{}^A\,. \tag{4.37}$$

The covariant transformations are also a useful tool to calculate any transformation on e.g. a quantity $W_A(q)\zeta^A$:

$$\begin{aligned}\delta\left(W_A(q)\zeta^A\right) &= \partial_X\left(W_A\zeta^A\right)\delta q^X + W_A\,\delta\zeta^A\big|_q\\ &= \mathfrak{D}_X\left(W_A\zeta^A\right)\delta q^X + W_A\left(\widehat{\delta}\zeta^A - \mathfrak{D}_X\zeta^A\delta q^X\right)\\ &= \left(\mathfrak{D}_X W_A\right)\delta q^X\zeta^A + W_A\,\widehat{\delta}\zeta^A\,.\end{aligned} \tag{4.38}$$

We will frequently use the covariant transformations (4.36).

Boson field equations. By varying the equations of motion of the fermions under supersymmetry, we derive the corresponding equations of motion for the scalar fields:

$$\widehat{\delta}(\epsilon)\Gamma^A = \frac{1}{2}\mathrm{i}f_X^{iA}\epsilon_i\Delta^X\,, \tag{4.39}$$

where

$$\Delta^X = \Box q^X - \frac{1}{2}\bar{\zeta}^B\gamma_a\zeta^D\partial^a q^Y f_Y^{iC}f_{iA}^X W_{BCD}{}^A - \frac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A\bar{\zeta}^E\zeta^D\bar{\zeta}^C\zeta^B f_E^{iY}f_{iA}^X\,, \tag{4.40}$$

and the covariant laplacian is given by

$$\Box q^X = \partial_a\partial^a q^X + \left(\partial_a q^Y\right)\left(\partial^a q^Z\right)\Gamma_{YZ}{}^X\,. \tag{4.41}$$

The equations of motion (4.33) and (4.40). These form a multiplet, as (4.39) has the counterpart

$$\widehat{\delta}(\epsilon)\Delta^X = -\mathrm{i}\bar{\epsilon}^i\not{\mathfrak{D}}\Gamma^A f_{iA}^X + 2\mathrm{i}\bar{\epsilon}^i\Gamma^B\bar{\zeta}^C\zeta^D f_{Bi}^Y\mathcal{R}^X{}_{YCD}\,, \tag{4.42}$$

where the covariant derivative of Γ^A is defined similar to (4.33).

Superconformal. To allow the generalization to superconformal couplings, the essential question is whether the manifold has dilatational symmetry. This means, according to (2.30) that there is a 'closed homothetic Killing vector' [50]. The dilatations act as[16]

$$\delta_D(\Lambda_D)q^X = \Lambda_D k_D^X(q), \tag{4.43}$$

where k_D^X satisfies (we generalize here already to d dimensions, as the modifications involve only a normalization factor)

$$\mathfrak{D}_Y k_D^X \equiv \partial_Y k_D^X + \Gamma^X_{YZ} k_D^Z = \frac{d-2}{2}\delta_Y{}^X. \tag{4.44}$$

The presence of this vector allows one to extend the transformations of rigid supersymmetry to the superconformal group [50, 51, 49], with e.g. transformations under the SU(2) R-symmetry group:

$$\delta_{\mathrm{SU(2)}}(\vec\Lambda)q^X = \vec\Lambda\cdot\vec k^X = \frac{2}{d-2}\vec\Lambda\cdot k_D^Y \vec J_Y{}^X\,. \tag{4.45}$$

On a flat manifold, where $D=\partial$, the fields q^X have Weyl weight 1. In general, one can introduce the sections

$$A^{iA} = k_D^X f_X^{iA}\,. \tag{4.46}$$

These transform under dilatations as

$$\widehat\delta_D(\Lambda_D)A^{iA} = \left(\partial_X A^{iA} - \omega_{XB}{}^A A^{iB}\right)\delta_D(\Lambda_D)q^X = \frac{d-2}{2}\Lambda_D A^{iA}\,. \tag{4.47}$$

They thus scale under dilatations as most of the fields we are used to with just a factor (weight), which is here $(d-2)/2$. E.g. in 5 dimensions, its transformation laws are

$$\widehat\delta A^{iB} = \frac{3}{2}f_X^{iB}\delta q^X = -\frac{3}{2}\mathrm{i}\bar\epsilon^i\zeta^B + \frac{3}{2}\Lambda_D A^{iB} - \Lambda^i{}_j A^{jB}. \tag{4.48}$$

We can then derive the other (super)conformal transformations using the algebra. The special conformal transformations on q^X and ζ^A vanish, apart from the induced parts as follows from (2.24). These implicit K-transformations imply e.g. that $\delta_K(\Lambda_K)\partial\!\!\!/ q^X\neq 0$. The algebra gives then for the S-supersymmetry [again apart from the implicit ones in $\epsilon(x)$]

$$\delta_S(\eta^i)\zeta^A = -k_D^X f_X^{iA}\eta_i\,. \tag{4.49}$$

The bosonic conformal symmetries act as

$$\widehat\delta_D\zeta^A = \frac{d-1}{2}\Lambda_D\zeta^A\,,\qquad \widehat\delta_{\mathrm{SU(2)}}\zeta^A = 0\,. \tag{4.50}$$

[16] Note that we give here only the intrinsic part of the dilatations, i.e. the Λ_D term in (2.24), and not the part included in the general coordinate transformation $\xi^\mu(x)$. Similarly for special conformal transformations, we will write here only the intrinsic part represented as $(k_\mu\phi)$ in that equation.

Isometries and coupling to vector multiplets. So far the multiplet was defined without a gauge group. Having in mind couplings to vector multiplets, one has to define the multiplet in the algebra including the vector multiplet with its gauge transformations. To do so, we first have to consider which transformations are possible. One may consider general transformations

$$\delta_{\rm G}(\alpha)q^X = -g\,\alpha^I k_I{}^X(q), \tag{4.51}$$

where g is the coupling constant and the $k_I{}^X(q)$ parametrize the transformations. When we have a metric, these vectors should be Killing vectors. As we have not discussed a metric yet, we could define here some generalization of symmetries, but we just refer the interested reader to [49]. The transformations constitute an algebra with structure constants $f_{IJ}{}^K$,

$$k_I{}^Y\partial_Y k_J{}^X - k_J{}^Y\partial_Y k_I{}^X = -f_{IJ}{}^K\,k_K{}^X\,. \tag{4.52}$$

These Killing vectors should respect the hypercomplex structure. This is the requirement vanishing of the commutator of $\mathfrak{D}_Y k_I^X$ with the complex structures:

$$\left(\mathfrak{D}_X k_I^Y\right)\vec{J}_Y{}^Z = \vec{J}_X{}^Y\left(\mathfrak{D}_Y k_I^Z\right)\,. \tag{4.53}$$

Extracting affine connections from this equation, it can be written as

$$\left(\mathcal{L}_{k_I}\vec{J}\right)_X{}^Y \equiv k_I^Z\partial_Z\vec{J}_X{}^Y - \partial_Z k_I^Y\,\vec{J}_X{}^Z + \partial_X k_I^Z\,\vec{J}_Z{}^Y = 0\,. \tag{4.54}$$

The left-hand side is the Lie derivative of the complex structure in the direction of the vector k_I. Note that these equations will be modified for quaternionic-Kähler manifolds. In that case, this is even not any more an independent equation, but follows from the fact that it is a Killing vector for a metric of a quaternionic-Kähler manifold. for hyper-Kähler manifolds, however, this is an extra constraint.

To define the transformations on the spinors one needs the matrix

$$t_{IA}{}^B = \frac{1}{2}f^Y_{iA}\mathfrak{D}_Y k_I^X f^{iB}_X. \tag{4.55}$$

The transformation of the fermions is then in 5 dimensions

$$\widehat{\delta}_G(\alpha)\zeta^A = -g\alpha^I t_{IB}{}^A(q)\zeta^B. \tag{4.56}$$

To illustrate the concept of the background of vector multiplets and Weyl multiplet for the hypermultiplet, we can give the full forms of (4.25), when the isometry with index I is coupled to the gauge symmetry of the vector field with index I:

$$\delta_Q(\epsilon)\zeta^A = \tfrac{1}{2}{\rm i}\not{D}q^X f^{iA}_X(q)\epsilon_i - \zeta^B\omega_{XB}{}^A(q)\left[\delta(\epsilon)q^X\right] + \frac{1}{3}\gamma^{ab}T_{ab}k^X f^{iA}_X\epsilon_i + \sqrt{\frac{3}{8\kappa^2}}h^I k_I^X f^{iA}_X\epsilon_i\,, \tag{4.57}$$

where T_{ab} is one of the auxiliary fields of the Weyl multiplet, and this term is thus absent in rigid supersymmetry. The covariant derivative includes now a term $A_\mu^I k_I^X$.

To 4 dimensions. In [52] it has been discussed how to translate the $d = 5$ results to a convenient $d = 4$ formulation. We consider the same bosonic fields q^X. The reduction to 4 dimensions is only nontrivial for the fermionic sector. It leads again to $2r$ spinors, whose right-handed part is ζ_A, with $A = 1, \ldots, 2r$ and the left-handed ones (C-conjugates of the former) are ζ^A. See the details in the appendix A.4. Thus, in absence of an SU(2) index on these spinors, the chirality is indicated by the fact that it has the index A up or down. One can start again by allowing arbitrary transformations for the scalars, and transformations of the spinors to derivatives of the scalars and deduce again the conditions on quantities that appear in these transformations. We would arrive again at (4.26) and (4.27). But as we have already done all the work for $d = 5$ (for which in fact the formalism is easier) we can also translate the results from what we already know.

We are lucky that the notations are compatible for bosonic quantities. The nontrivial issue here is that in 4 dimensions we raise and lower indices by complex conjugation, while in 5 dimensions we used the raising and lowering by means of the matrices ρ^{AB} and ρ_{AB}. For the fermionic part we need some translation rules. Using as in appendix A.4 a tilde for the 5-dimensional spinors, we have

$$\begin{aligned} \zeta^A &= P_L\tilde{\zeta}^A\,, & \zeta_A &= P_R\tilde{\zeta}^B\rho_{BA} = P_R\tilde{\zeta}_A\,,\\ \epsilon^i &= P_L\tilde{\epsilon}^i\,, & \epsilon_i &= P_R\tilde{\epsilon}^j\varepsilon_{ji} = P_R\tilde{\epsilon}_i\,. \end{aligned} \tag{4.58}$$

This leads in 4 dimensions to[17]

$$\begin{aligned} \delta_Q(\epsilon)q^X &= -\mathrm{i}f^X_{iA}\bar{\epsilon}^i\zeta^A + \mathrm{i}f^{XiA}\bar{\epsilon}_i\zeta_A,\\ \delta_Q(\epsilon)\zeta^A &= \tfrac{1}{2}\mathrm{i}f^{iA}_X\not{\partial}q^X\epsilon_i - \zeta^B\omega_{XB}{}^A\delta_Q(\epsilon)q^X,\\ \delta_Q(\epsilon)\zeta_A &= -\tfrac{1}{2}\mathrm{i}f_{XiA}\not{\partial}q^X\epsilon^i + \zeta_B\omega_{XA}{}^B\delta_Q(\epsilon)q^X. \end{aligned} \tag{4.59}$$

We use here the notation[18]

$$f^{XiA} \equiv \varepsilon^{ij}\rho^{AB}f^X_{jB} = (f^X_{iA})^*. \tag{4.60}$$

Remark how charge conjugation changes the position of SU(2) indices i and USp indices A, but leaves the index X referring to the real scalar coordinates invariant. The object $\omega_{XA}{}^B$ is, however, 'imaginary' under this conjugation [see (A.31)], which leads to the sign difference between the last terms of the supersymmetry variations of the chiral and antichiral fermions.

Some formulae for the fermions differ because of the chiral notations. We obtain the supersymmetry algebra

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]\,\zeta^A = \xi^\mu\partial_\mu\zeta^A - \tfrac{1}{2}\varepsilon^{ij}\rho^{AB}\Gamma_B\bar{\epsilon}_{1i}\epsilon_{2j} + \tfrac{1}{2}\gamma_\mu\bar{\epsilon}^i_{[1}\gamma^\mu\epsilon_{2]i}\Gamma^A\,, \tag{4.61}$$

[17] Remark that a Majorana conjugated spinor of five dimensions gets due to (A.68) an extra factor γ_5 in 4 dimensions.

[18] Note that this notation is different from [49, 53] where C_{AB}, the matrix introduced in (4.87), is used for raising and lowering symplectic indices. The conventions differ only for the case of non-positive definite metrics. The convention that we take here is more appropriate for 4 dimensions as it allows us to keep the rule that charge conjugation raises and lowers indices.

with ξ^μ as in (2.6). The fermion field equation is now

$$\begin{aligned}\Gamma^A &\equiv \mathfrak{D}\zeta^A - \tfrac{1}{2}W_{BC}{}^{DA}\zeta_D\bar{\zeta}^B\zeta^C\,,\\ \Gamma_A &\equiv \mathfrak{D}\zeta_A - \tfrac{1}{2}W^{BC}{}_{DA}\bar{\zeta}_B\zeta_C\zeta^D\,,\end{aligned} \tag{4.62}$$

where e.g. $W_{BC}{}^{DA} = \rho^{DE}W_{BCE}{}^A$ and $W^{BC}{}_{DA}$ is its complex conjugate, and the translation for the field equation from $d = 5$ (tildes here) to $d = 4$ goes with

$$\Gamma^A = P_R\tilde{\Gamma}^A\,, \qquad \Gamma_A = P_L\tilde{\Gamma}^B\rho_{BA} = P_L\tilde{\Gamma}_A\,. \tag{4.63}$$

Exercise 4.6: Derive these expressions from (4.33) using the projections as defined in (A.69) and (A.70). To translate $\bar{\zeta}^C\zeta^D$ from 5 dimensions to 4, one has to split it in the two chiralities. Furthermore, one has to use that the charge conjugation matrices differ by a factor γ_5, and finally that Fierz identities in 4 dimensions [see (A.64)] imply that the symmetric part $\zeta_{(B}\bar{\zeta}_C\zeta_{D)}$ vanishes.

Also the treatment of symmetries can be taken over from 5 dimensions. For the conformal symmetries, we have to consider also the U(1) in the superconformal group. The scalars are inert under this transformation, while the fermions have

$$\begin{aligned}\widehat{\delta}_D\zeta^A &= \tfrac{3}{2}\zeta^A,\\ \widehat{\delta}_{\mathrm{U}(1)}\zeta^A &= \tfrac{1}{2}\mathrm{i}\zeta^A,\\ \widehat{\delta}_{\mathrm{SU}(2)}\zeta^A &= 0.\end{aligned} \tag{4.64}$$

4.2.4 Tensor multiplet in $d = 4$

The tensor multiplet in $d = 4$ dimensions was obtained in [54]. We can immediately give now the transformation rules in the background of conformal supergravity:

$$\begin{aligned}\delta L_{ij} &= \bar{\epsilon}_{(i}\varphi_{j)} + \varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\varphi^{l)}\,,\\ \delta\varphi^i &= \tfrac{1}{2}\not{D}L^{ij}\,\epsilon_j + \tfrac{1}{2}\varepsilon^{ij}\,\hat{\not{E}}^I\,\epsilon_j - \tfrac{1}{2}G\,\epsilon^i + 2L^{ij}\,\eta_j\,,\\ \delta G &= -\,\bar{\epsilon}_i\not{D}\varphi^i - \bar{\epsilon}_i(3L^{ij}\,\chi_j - \tfrac{1}{8}\gamma^{ab}T^+_{ab}\varphi_j\varepsilon^{ij}\varepsilon^{ij}) + 2\bar{\eta}_i\varphi^i\,,\\ \delta E_{\mu\nu} &= \tfrac{1}{2}\mathrm{i}\bar{\epsilon}^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - \tfrac{1}{2}\mathrm{i}\bar{\epsilon}_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij} + \mathrm{i}L_{ij}\varepsilon^{jk}\bar{\epsilon}^i\gamma_{[\mu}\psi_{\nu]k} - \mathrm{i}L^{ij}\varepsilon_{jk}\bar{\epsilon}_i\gamma_{[\mu}\psi_{\nu]}{}^k\,.\end{aligned} \tag{4.65}$$

Obviously, the result in rigid supersymmetry is the above one where the fields of the Weyl multiplet, T and ψ_μ, has been put equal to zero, and the covariant derivatives are replaced by ordinary derivatives.

A first step in building actions from this multiplet has been set in [54], but more applications can be found in [55].

4.3 Construction of the superconformal actions.

We have now the building stones to proceed to constructions of general actions. First, I will review how the conformal calculus works in a simple bosonic example. Then I can go through the full construction of actions in the superconformal framework and their gauge fixing to super-Poincaré theories.

4.3.1 Vector multiplets in 4 dimensions

As explained in section 4.1, the idea is to start by constructing an action invariant under superconformal group. Later, one chooses gauges for the extra gauge invariances of the superconformal group, such that the remaining theory has just the super-Poincaré symmetries.

The 'R-symmetry group' $\mathrm{SU}(2) \times \mathrm{U}(1)$ plays an important role:

- the gauge connection of $U(1)$ will be the Kähler curvature. It acts on the manifold of scalars in vector multiplets,
- the gauge connection of $\mathrm{SU}(2)$ promotes the hyper-Kähler manifold of hypermultiplets to a quaternionic manifold.

Neglecting for now the hypermultiplets, we have to consider the basic supergravity multiplet and the vector multiplets. The physical content that one should have (from representation theory of the super-Poincaré group) can be represented as follows:

$$\begin{array}{cccccccc}
 & \text{SUGRA} & & & & \text{vector multiplet} & & \\
 & 2 & & & & & & \\
\frac{3}{2} & & \frac{3}{2} & & & & & \\
 & 1 & & & & 1 & & \to n+1 \\
 & & & +n\,* & \frac{1}{2} & & \frac{1}{2} & \\
 & & & & 0 & & 0 &
\end{array} \qquad (4.66)$$

The supergravity sector contains the graviton, 2 gravitini and a so-called graviphoton. That spin-1 field[19] gets, by coupling to n vector multiplets, part of a set of $n+1$ vectors, which will be uniformly described by the special Kähler geometry. The scalars appear as n complex ones z^α, with $\alpha = 1, \ldots, n$.

To describe this, the idea is to start with $n+1$ superconformal vector multiplets with scalars X^I with $I = 0, \ldots, n$. One of these multiplets should then be the compensating one. Its vector survives, but the missing fermions and scalars are the ones that have been used to fix superfluous gauge symmetries of the superconformal algebra.

Remembering that vector multiplets are constrained chiral supermultiplets, we can form new chiral superfields by taking an arbitrary holomorphic function of the vector multiplets.

[19] We use here and below freely the terminology 'spin 1' for vectors, spin-$\frac{1}{2}$ for spinors, ..., though of course only in 4 dimensions the representations of the little group of the Lorentz group can be characterized by just one number, which is called 'spin'. In higher dimensions, the representations should be characterized by more numbers, but often the same fields, like graviton as a symmetric tensor, vectors, ... occur, and we denote them freely with the terminology that is appropriate for the 4-dimensional fields.

Consider the chiral multiplet with lowest component $A=\frac{1}{2}\mathrm{i}F(X)$ (the overall normalization is for later convenience to get a result with the normalization that is currently most used in the literature). The further components are then defined by the transformation law, which gives, comparing with (4.13), $\Psi_i=\frac{1}{2}\mathrm{i}F_I\Omega_i^I$, where I use here and below the notation

$$\begin{aligned} F_I(X) &= \frac{\partial}{\partial X^I}F(X)\,, \qquad \bar F_I(\bar X) = \frac{\partial}{\partial \bar X^I}\bar F(\bar X)\,,\\ F_{IJ} &= \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}F(X) \qquad \ldots . \end{aligned} \tag{4.67}$$

Calculating the transformation of Ψ_i one finds B_{ij} and F_{ab}, ... See

$$\begin{aligned} A &= \tfrac{1}{2}\mathrm{i}F\\ \Psi_i &= \tfrac{1}{2}\mathrm{i}F_I\Omega_i^I\\ B_{ij} &= \tfrac{1}{2}\mathrm{i}F_IY_{ij}^I-\tfrac{1}{4}\mathrm{i}F_{IJ}\bar\Omega_i^I\Omega_j^J\\ G_{ab}^- &= \tfrac{1}{2}\mathrm{i}F_I\mathcal{F}_{ab}^{-I}-\tfrac{1}{16}\mathrm{i}F_{IJ}\bar\Omega_i^I\gamma_{ab}\Omega_j^J\varepsilon^{ij}\\ \Lambda_i &= -\tfrac{1}{2}\mathrm{i}F_I\not{D}\Omega^{jI}\varepsilon_{ij}-\tfrac{1}{2}\mathrm{i}gF_If_{JK}^I\bar X^J\Omega_i^K-\tfrac{1}{8}\mathrm{i}F_{IJ}\gamma^{ab}\mathcal{F}_{ab}^{-I}\Omega_i^J\\ &\quad -\tfrac{1}{4}\mathrm{i}F_{IJ}\Omega_k^JY_{ij}^I\varepsilon^{jk}+\tfrac{1}{96}\mathrm{i}F_{IJK}\gamma^{ab}\Omega_i^I\bar\Omega_j^J\gamma_{ab}\Omega_k^K\varepsilon^{jk}\\ C &= -\mathrm{i}F_ID_aD^a\bar X^I-\tfrac{1}{4}\mathrm{i}F_I\mathcal{F}_{ab}^{+I}T^{+ab}-\tfrac{3}{2}\mathrm{i}F_I\bar\chi_i\Omega^{iI}+\tfrac{1}{2}\mathrm{i}gF_If_{JK}^I\bar\Omega^{iJ}\Omega^{jK}\varepsilon_{ij}\\ &\quad -\mathrm{i}g^2F_If_{JK}^If_{LM}^J\bar X^K\bar X^LX^M-\tfrac{1}{8}\mathrm{i}F_{IJ}Y^{ijI}Y_{ij}^J+\tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}_{ab}^{-I}\mathcal{F}^{-abJ}\\ &\quad +\tfrac{1}{2}\mathrm{i}F_{IJ}\bar\Omega_i^I\not{D}\Omega^{iJ}-\tfrac{1}{2}\mathrm{i}gF_{IJ}f_{KL}^I\bar X^K\bar\Omega_i^J\Omega_j^L\varepsilon^{ij}+\tfrac{1}{8}\mathrm{i}F_{IJK}Y^{ijI}\bar\Omega_i^J\Omega_j^K\\ &\quad -\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar\Omega_i^I\gamma^{ab}\mathcal{F}_{ab}^{-J}\Omega_j^K+\tfrac{1}{48}\mathrm{i}F_{IJKL}\bar\Omega_i^I\Omega_l^J\bar\Omega_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{kl}. \end{aligned} \tag{4.68}$$

The action is determined by the components of this multiplet as I symbolically wrote in (4.21). For rigid supersymmetry, it is just the highest component of the multiplet, C. In (conformal) supergravity one needs further 'Noether terms' in order that the transformation is also a total derivative when the supersymmetry parameter is a local function. Moreover, as already mentioned at the end of section 4.2.2, one needs a chiral multiplet with $w=2$, e.g. in order that C does not transform under the U(1), see (4.18), and that the dilatation invariance is respected. Furthermore, the last terms in (4.19) have to be compensated. Therefore, the 'density formula' is in this case [56]

$$\begin{aligned} e^{-1}\mathcal{L} &= C-\bar\psi_i\cdot\gamma\Lambda_j\varepsilon^{ij}+\frac{1}{8}\bar\psi_{\mu i}\gamma\cdot T^+\gamma^\mu\Psi_j\varepsilon^{ij}-\frac{1}{4}AT_{ab}^+T^{+ab}-\frac{1}{2}\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}B_{kl}\varepsilon^{ik}\varepsilon^{jl}\\ &\quad +\bar\psi_{\mu i}\psi_{\nu j}\varepsilon^{ij}G^{-\mu\nu}-\bar\psi_{\mu i}\psi_{\nu j}\varepsilon^{ij}AT^{+\mu\nu}+\frac{1}{2}\mathrm{i}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\gamma_\sigma\Psi_l\\ &\quad +\frac{1}{2}\mathrm{i}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\psi_{\sigma l}A+\text{h.c.} \end{aligned} \tag{4.69}$$

In order to get the Weyl weight $w=2$ for the chiral multiplet, $F(X)$ to be homogeneous of weight 2, where the X fields carry weight 1. This implies the following relations for the derivatives of F:

$$2F=F_IX^I\,,\qquad F_{IJ}X^J=F_I\,,\qquad F_{IJK}X^K=0. \tag{4.70}$$

Inserting then (4.68) in (4.69) leads to

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& -\mathrm{i}F_I D_a D^a \bar{X}^I + \tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\mathcal{F}^{-abJ} + \tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}^I_i \not{D}\Omega^{iJ}\\
&-\tfrac{1}{8}\mathrm{i}F_{IJ}Y^{ijI}Y^J_{ij} + \tfrac{1}{8}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}^J_i\Omega^K_j\\
&-\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}^I_i\gamma^{ab}\mathcal{F}^{-J}_{ab}\Omega^K_j + \tfrac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}^I_i\Omega^J_\ell\bar{\Omega}^K_j\Omega^L_k\varepsilon^{ij}\varepsilon^{k\ell}\\
&+\tfrac{1}{2}\mathrm{i}gF_I f_{JK}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij} - \tfrac{1}{2}\mathrm{i}gF_{IJ}f^I_{KL}\bar{X}^K\bar{\Omega}^J_i\Omega^L_j\varepsilon^{ij} - \mathrm{i}g^2F_If^I_{JK}f^J_{LM}\bar{X}^K\bar{X}^LX^M\\
&-\tfrac{1}{4}\mathrm{i}F_I\mathcal{F}^{+I}_{ab}T^{+ab} - \tfrac{3}{2}\mathrm{i}F_I\bar{\chi}_i\Omega^{iI} - \tfrac{1}{4}\mathrm{i}F_{IJ}\bar{\psi}_i\cdot\gamma\Omega^I_jY^{ijJ}\\
&+\tfrac{1}{2}\mathrm{i}gF_If^I_{JK}\bar{X}^J\bar{\psi}_i\cdot\gamma\Omega^K_j\varepsilon^{ij} - \tfrac{1}{2}\mathrm{i}F_I\bar{\psi}_i\cdot\gamma\not{D}\Omega^{iI} + \tfrac{1}{8}\mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\bar{\psi}_i\cdot\gamma\gamma^{ab}\Omega^J_j\varepsilon^{ij}\\
&+\tfrac{1}{12}\mathrm{i}F_{IJK}\bar{\Omega}^J_l\Omega^K_j\bar{\psi}_i\cdot\gamma\Omega^I_k\varepsilon^{ij}\varepsilon^{kl} + \tfrac{1}{16}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma\cdot T^+\gamma^\mu\Omega^I_j\varepsilon^{ij} - \tfrac{1}{8}\mathrm{i}FT^+_{ab}T^{+ab}\\
&-\tfrac{1}{4}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}Y^{ijI} - \tfrac{1}{2}\mathrm{i}FT^{+\mu\nu}\bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij} + \tfrac{1}{2}\mathrm{i}F_I\mathcal{F}^{-\mu\nu I}\bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij}\\
&-\tfrac{1}{16}\mathrm{i}F_{IJ}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\Omega}^I_k\gamma^{\mu\nu}\Omega^J_\ell\varepsilon^{ij}\varepsilon^{k\ell} + \tfrac{1}{8}\mathrm{i}F_{IJ}\bar{\Omega}^I_k\Omega^J_l\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\varepsilon^{ik}\varepsilon^{jl}\\
&-\tfrac{1}{4}F_I\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\gamma_\sigma\Omega^I_\ell - \tfrac{1}{4}F\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\psi_{\sigma\ell}\\
&+\text{h.c.}.
\end{aligned}
\tag{4.71}
$$

The first terms of the action (4.71) are kinetic terms for the scalars X, the vectors, and the fermions Ω. The following term says that Y_{ij} is an auxiliary field that can be eliminated by its field equation. The first 4 lines are the ones that we would encounter also in rigid supersymmetry, see (E.1). For these terms, the relations (4.70) have not been used, and this part is thus the general result for rigid supersymmetry. The other lines are due to the local superconformal symmetry. For those interested in rigid symmetry, I repeat that in that case the covariant derivatives reduce to e.g.

$$
\begin{aligned}
D_aX^I &= \partial_aX^I - gW^K_aX^Jf_{JK}{}^I,\\
D_a\Omega^I_i &= \partial_aX^I - gW^K_a\Omega^J_if_{JK}{}^I,\\
\mathcal{F}^I_{ab} &= 2\partial_{[a}W^I_{b]} + gW^K_bW^J_af_{JK}{}^I.
\end{aligned}
\tag{4.72}
$$

For the gauging I assumed here that F is an invariant function, i.e.

$$
\delta_GF \equiv gF_I\alpha^KX^Jf_{JK}{}^I = 0, \tag{4.73}
$$

although one may allow that F transforms in a quadratic function with real coefficients [56, 57]:

$$
\delta_GF = -g\alpha^IC_{I,JK}X^JX^K, \tag{4.74}
$$

where $C_{I,JK}$ are real constants that enter then in an additional Chern–Simons term. We will come back to this possibility in section 6.5. The statement of invariance of F leads, after one derivative to X that

$$
F_If_{JK}{}^I = -F_{IJ}f_{LK}{}^IX^L. \tag{4.75}
$$

Simplifications In order to get a more useful form of the action, one has to make the conformal covariant derivatives explicit. The principle is explained for the bosonic case in

(4.2). This leads here to

$$\begin{aligned}\Box^C \bar X^I &= \widehat{\partial}_\mu D^\mu \bar X^I - \omega_\mu{}^{\mu\nu} D_\nu \bar X^I - \mathrm{i} A_\mu D^\mu \bar X^I + 2 f_\mu{}^\mu \bar X^I - \tfrac12 \bar\psi_{\mu i} D^\mu \Omega^{iI}\\ &\quad + \tfrac{1}{32}\bar\psi^i_\mu \gamma^\mu \gamma\cdot T^+ \Omega^{jI}\varepsilon_{ij} - \tfrac12 \bar\Omega^{iI}\gamma\cdot\phi_i - \tfrac34 \bar\psi^i\cdot\gamma\chi_i \bar X^I\\ &\quad -\tfrac12 g\varepsilon^{ij}\bar\psi_i\cdot\gamma\Omega^J_j \bar X^K f_{JK}^I - \tfrac12 g\varepsilon_{ij}\bar\psi^i\cdot\gamma\Omega^{jJ}\bar X^K f_{JK}^I. \end{aligned} \tag{4.76}$$

This leads to the expression for the first term of (4.71)

$$\begin{aligned} -\mathrm{i}eF_I\Box^C\bar X^I &= \mathrm{i}eF_{IJ}\mathcal{D}_\mu X^I \mathcal{D}^\mu \bar X^J - \tfrac12 \mathrm{i}eF_{IJ}\mathcal{D}_\mu X^J \bar\psi^\mu_i \Omega^{iI} - 2\mathrm{i}eF_I f_\mu{}^\mu \bar X^I + \tfrac12 \mathrm{i}eF_I\bar\psi_{\mu i}D^\mu\Omega^{iI}\\ &\quad -\tfrac{1}{32}\mathrm{i}eF_I\bar\psi^i_\mu\gamma^\mu\gamma\cdot T^- \tfrac12 \Omega^{jI}\varepsilon_{ij} + \tfrac12 \mathrm{i}eF_I\bar\Omega^{iI}\gamma\cdot\phi_i + \tfrac34 \mathrm{i}eF_I\bar\psi^i\cdot\gamma\chi_i\bar X^I\\ &\quad +\tfrac12 g\mathrm{i}eF_I\varepsilon^{ij}\bar\psi_i\cdot\gamma\Omega_j^J\bar X^K f_{JK}^I + \tfrac12 g\mathrm{i}eF_I\varepsilon_{ij}\bar\psi^i\cdot\gamma\Omega^{jJ}\bar X^K f_{JK}^I\\ &\quad +\mathrm{i}eF_I\bar\psi^i_{[\mu}\gamma^\nu\psi_{\nu]i}\mathcal{D}^\mu\bar X^I - \tfrac12 \mathrm{i}eF_I\bar\psi^i_{[\mu}\gamma^\nu\psi_{\nu]i}\bar\psi^\mu_i\Omega^{iI} + \text{ total derivative.}\end{aligned}\tag{4.77}$$

The other term that has to be written explicitly is the covariant derivative of the fermions. This leads to

$$\begin{aligned}\tfrac12 \mathrm{i}eF_{IJ}\bar\Omega^I_i \not{D}\Omega^{iJ} &= \tfrac12 \mathrm{i}eF_{IJ}\bar\Omega^I_i \not{\mathcal{D}}\Omega^{iJ} - \tfrac12 \mathrm{i}eF_{IJ}\bar\Omega^I_i\gamma^\mu\gamma^\nu\psi^i_\mu\mathcal{D}_\nu\bar X^J + \tfrac14 \mathrm{i}eF_{IJ}\bar\Omega^I_i\gamma^\mu\gamma^\nu\psi^i_\mu\bar\psi_{\nu j}\Omega^{jJ}\\ &\quad -\tfrac14 \mathrm{i}eF_{IJ}\bar\Omega^I_i\gamma\cdot\psi_j Y^{ijJ} - \tfrac18 \mathrm{i}eF_{IJ}\bar\Omega^I_i\gamma_\mu\gamma\cdot\mathcal{F}^{+J}\psi^\mu_j\varepsilon^{ij}\\ &\quad -\tfrac12 g\mathrm{i}eF_{IJ}\bar\Omega^I_i\gamma\cdot\psi_j\varepsilon^{ij}\bar X^K X^L f^J_{KL} - \mathrm{i}eF_{IJ}\bar X^J\bar\Omega^I_i\gamma\cdot\phi^i.\end{aligned}\tag{4.78}$$

Deleting total derivatives, the action is at this point

$$\begin{aligned} e^{-1}\mathcal{L} &= \mathrm{i}F_{IJ}\mathcal{D}_\mu X^I\mathcal{D}^\mu\bar X^J - 2\mathrm{i}F_I f_\mu{}^\mu\bar X^I + \tfrac14 \mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\mathcal{F}^{-abJ} - \tfrac18 \mathrm{i}FT^+_{ab}T^{+ab} - \tfrac14 \mathrm{i}F_I\mathcal{F}^{+I}_{ab}T^{+ab}\\ &\quad -\tfrac18 \mathrm{i}F_{IJ}Y^{ijI}Y^J_{ij} - \mathrm{i}g^2 F_I f^I_{JK}f^J_{LM}\bar X^K\bar X^L X^M\\ &\quad +\mathrm{i}F_I\bar X^I\bar\psi_{\mu i}\gamma^{\mu\nu}\phi^i_\nu + \tfrac12 \mathrm{i}F_{IJ}\bar\Omega^I_i\not{\mathcal{D}}\Omega^{iJ} + \tfrac12 \mathrm{i}F_I\bar\psi_{\mu i}\gamma^{\mu\nu}\gamma^\rho\mathcal{D}_\rho\bar X^I\psi^i_\nu + \tfrac12 \mathrm{i}F_I\bar\Omega^{iI}\gamma\cdot\phi_i\\ &\quad +\tfrac18 \mathrm{i}F_I\bar\psi_{\mu i}\gamma^{\mu\nu}\gamma\cdot\mathcal{F}^{+I}\varepsilon^{ij}\psi_{\nu j} - \tfrac{1}{32}\mathrm{i}F_I\bar\psi^i_\mu\gamma^\mu\gamma\cdot T^+\Omega^{jI}\varepsilon_{ij} + \tfrac34 \mathrm{i}F_I\bar\psi^i\cdot\gamma\chi_i\bar X^I\\ &\quad +\tfrac12 g\mathrm{i}F_I\varepsilon^{ij}\bar\psi_i\cdot\gamma\Omega^J_j\bar X^K f^I_{JK} + \tfrac12 g\mathrm{i}F_I\varepsilon_{ij}\bar\psi^i\cdot\gamma\Omega^{jJ}\bar X^K f^I_{JK} + \mathrm{i}F_I\bar\psi^i_{[\mu}\gamma^\nu\psi_{\nu]i}\mathcal{D}^\mu\bar X^I\\ &\quad -\tfrac12 \mathrm{i}F_I\bar\psi^i_{[\mu}\gamma^\nu\psi_{\nu]i}\bar\psi^\mu_i\Omega^{iI} - \tfrac32 \mathrm{i}F_I\bar\chi_i\Omega^{iI} + \tfrac12 \mathrm{i}gF_I f^I_{JK}\bar\Omega^{iJ}\Omega^{jK}\varepsilon_{ij}\\ &\quad -\tfrac12 \mathrm{i}F_{IJ}\mathcal{D}_\mu X^J\bar\psi^\mu_i\Omega^{iI} - \tfrac12 \mathrm{i}F_I\bar\psi_{\mu i}\gamma^{\mu\nu}\mathcal{D}_\nu\Omega^{iI}\\ &\quad -\tfrac12 \mathrm{i}F_{IJ}\bar\Omega^I_i\gamma^\mu\gamma^\nu\psi^i_\mu\mathcal{D}_\nu\bar X^J + \tfrac12 \mathrm{i}F_I\mathcal{F}^{-\mu\nu I}\bar\psi_{\mu i}\psi_{\nu j}\varepsilon^{ij}\\ &\quad -\tfrac18 \mathrm{i}F_{IJ}\bar\Omega^I_i\gamma_\mu\gamma\cdot\mathcal{F}^{+J}\psi^\mu_j\varepsilon^{ij} - \tfrac12 g\mathrm{i}F_{IJ}\bar\Omega^I_i\gamma\cdot\psi_j\varepsilon^{ij}\bar X^K X^L f^J_{KL}\\ &\quad -\mathrm{i}F_{IJ}\bar X^J\bar\Omega^I_i\gamma\cdot\phi^i - \tfrac12 \mathrm{i}gF_{IJ}f^I_{KL}\bar X^K\bar\Omega^J_i\Omega^L_j\varepsilon^{ij} + \tfrac18 \mathrm{i}F_{IJK}Y^{ijI}\bar\Omega^J_i\Omega^K_j\\ &\quad -\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar\Omega^I_i\gamma^{ab}\mathcal{F}^{-J}_{ab}\Omega^K_j - \tfrac12 \mathrm{i}FT^{+\mu\nu}\bar\psi_{\mu i}\psi_{\nu j}\varepsilon^{ij}\\ &\quad +\tfrac12 \mathrm{i}gF_I f^I_{JK}\bar X^J\bar\psi_i\cdot\gamma\Omega^K_j\varepsilon^{ij} + \tfrac18 \mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\bar\psi_i\cdot\gamma\gamma^{ab}\Omega^J_j\varepsilon^{ij} + \tfrac{1}{16}\mathrm{i}F_I\bar\psi_{\mu i}\gamma\cdot T^+\gamma^\mu\Omega^I_j\varepsilon^{ij}\\ &\quad +\tfrac{1}{12}\mathrm{i}F_{IJK}\bar\Omega^J_\ell\Omega^K_j\bar\psi_i\cdot\gamma\Omega^I_k\varepsilon^{ij}\varepsilon^{k\ell} - \tfrac14 \mathrm{i}F_I\bar\psi_{\mu i}\gamma^{\mu\nu}\gamma^\rho\psi^i_\nu\bar\psi_{\rho k}\Omega^{kI}\\ &\quad +\tfrac18 \mathrm{i}F_{IJ}\bar\Omega^I_k\Omega^J_\ell\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\varepsilon^{ik}\varepsilon^{j\ell} + \tfrac14 \mathrm{i}F_{IJ}\bar\Omega^I_i\gamma^\mu\gamma^\nu\psi^i_\mu\bar\psi_{\nu j}\Omega^{jJ}\\ &\quad +\tfrac{1}{48}\mathrm{i}F_{IJKL}\bar\Omega^I_i\Omega^J_\ell\bar\Omega^K_j\Omega^L_k\varepsilon^{ij}\varepsilon^{k\ell} - \tfrac{1}{16}\mathrm{i}F_{IJ}\bar\psi_{\mu i}\psi_{\nu j}\bar\Omega^I_k\gamma^{\mu\nu}\Omega^J_\ell\varepsilon^{ij}\varepsilon^{k\ell}\\ &\quad -\tfrac14 F_I\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\gamma_\sigma\Omega^I_\ell - \tfrac14 F\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\psi_{\sigma\ell} + \text{h.c.}\end{aligned}\tag{4.79}$$

We use then (4.11) and the values of the conformal gauge fields that follow from the constraints:

$$\begin{aligned} f_\mu{}^\mu &= -\tfrac{1}{12}R - \tfrac{1}{2}D + \left\{\tfrac{1}{8}\bar\psi^i\cdot\gamma\chi_i + \tfrac{1}{24}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi^i_\mu\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} + \tfrac{1}{24}\bar\psi^i_\mu\psi^j_\nu\varepsilon_{ij}T^{+\mu\nu} + \text{h.c.}\right\}, \\ \phi^i_\mu &= \tfrac{1}{4}\gamma_\mu\chi^i + \tfrac{1}{4}\left(\gamma^{\nu\rho}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\nu\rho}\right)\left(\mathcal{D}_\nu\psi^i_\rho - \tfrac{1}{16}\gamma\cdot T^-\varepsilon^{ij}\gamma_\nu\psi_{\rho j}\right). \end{aligned} \tag{4.80}$$

This leads to various simplifications. The leading terms of the Lagrangian are then

$$\begin{aligned} e^{-1}\mathcal{L} = &-\tfrac{1}{6}RN_{IJ}\bar X^I X^J - DN_{IJ}\bar X^I X^J - N_{IJ}\mathcal{D}_\mu X^I\mathcal{D}^\mu\bar X^J + \tfrac{1}{8}N_{IJ}Y^{ijI}Y^J_{ij} \\ &+g^2N_{IJ}f^I_{KL}\bar X^K X^L f^J_{MN}\bar X^M X^N \\ &+\Big\{-\tfrac{1}{4}\mathrm{i}\bar F_{IJ}\widehat F^{+I}_{\mu\nu}\widehat F^{+\mu\nu J} - \tfrac{1}{4}N_{IJ}\bar\Omega^{iI}\not{\mathcal{D}}\Omega^J_i - \tfrac{1}{2}N_{IJ}\bar X^I X^J\bar\psi_i\cdot\gamma\chi^i \\ &+N_{IJ}X^I\bar\chi_i\Omega^{iJ} - \tfrac{1}{3}N_{IJ}\bar X^J\bar\Omega^I_i\gamma^{\mu\nu}\mathcal{D}_\mu\psi^i_\nu + \tfrac{1}{2}N_{IJ}\bar\psi^i_\mu\not{\mathcal{D}}\bar X^I\gamma^\mu\Omega^J_i \\ &-\tfrac{1}{6}\mathrm{i}N_{IJ}\bar X^I X^J e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\gamma_\nu\mathcal{D}_\rho\psi^i_\sigma \\ &+\tfrac{1}{4}\mathrm{i}N_{IJ}\bar X^I e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\gamma_\nu\psi^i_\rho\mathcal{D}_\sigma X^J + \tfrac{1}{12}N_{IJ}\bar X^I X^J\bar\psi^a_i\psi^b_j T^-_{ab}\varepsilon^{ij} \\ &-\tfrac{1}{16}N_{IJ}X^I X^J T^+_{ab}T^{+ab} + \tfrac{1}{4}N_{IJ}X^I\widehat F^{+J}_{ab}T^{+ab} \\ &-\tfrac{1}{6}N_{IJ}\bar X^I\bar\Omega^J_i\gamma^a\psi^b_j T^-_{ab}\varepsilon^{ij} - \tfrac{1}{32}\mathrm{i}F_{IJK}\bar\Omega^I_i\gamma^{ab}\Omega^J_j\bar X^K T^-_{ab}\varepsilon^{ij} \\ &+\tfrac{1}{4}F_{IJK}\mathcal{D}_\mu X^I\bar\Omega^J_i\gamma^\mu\Omega^{iK} + \tfrac{1}{8}\mathrm{i}F_{IJK}Y^{ijI}\bar\Omega^J_i\Omega^K_j + \text{h.c.}\Big\} + \cdots, \end{aligned} \tag{4.81}$$

where we introduced

$$N_{IJ} \equiv 2\,\mathrm{Im}\,F_{IJ} = -\mathrm{i}F_{IJ} + \mathrm{i}\bar F_{IJ}. \tag{4.82}$$

We also repeat the meaning of the covariant derivatives

$$\begin{aligned} \mathcal{D}_a X^I &\equiv (\partial_a - b_a + \mathrm{i}A_a)X^I + gf^I_{JK}W^J_a X^K, \\ \mathcal{D}_a\Omega^I_i &\equiv \left(\partial_a + \tfrac{1}{4}\omega_a{}^{bc}\gamma_{bc} - \tfrac{3}{2}b_a + \tfrac{1}{2}\mathrm{i}A_a\right)\Omega^I_i + V_{ai}{}^j\Omega^I_j + gf^I_{JK}W^J_a\Omega^K_i, \\ \widehat F_{\mu\nu}{}^I &= F_{\mu\nu}{}^I + \left(-\varepsilon_{ij}\bar\psi^i_{[\mu}\gamma_{\nu]}\Omega^{Ij} - 2\varepsilon_{ij}\bar\psi^i_\mu\psi^j_\nu\bar X^I + \text{h.c.}\right), \\ F_{\mu\nu}{}^I &= \partial_\mu W_\nu - \partial_\nu W_\mu + gW^J_\mu W^K_\nu f_{JK}{}^I. \end{aligned} \tag{4.83}$$

4.3.2 Vector multiplets in 5 dimensions

XXX $d=5$ rigid has only quadratic and cubic terms, see [58]. XXX

4.3.3 Hypermultiplets

While the actions of vector multiplets were constructed using tensor calculus manipulations, for the hypermultiplets we use another procedure. The main difference is that we have already the field equations in this case. We thus want to construct an action such that these field equations are obtained as Euler-Lagrange equations of the action. E.g. (in 5 dimensions) we define the non-closure functions Γ^A to be proportional to the field equations for the fermions ζ^A as

$$\frac{\delta S_{\text{hyper}}}{\delta\bar\zeta^A} = 2C_{AB}\Gamma^B\,. \tag{4.84}$$

In general, the tensor C_{AB} could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in AB and

$$\frac{\delta C_{AB}}{\delta \zeta^C} = 0\,, \tag{4.85}$$

$$\mathfrak{D}_X C_{AB} = 0\,. \tag{4.86}$$

This means that the tensor does not depend on the fermions and is covariantly constant.

We then identify also the bosonic field equations. It will turn out that the signature is determined by the Hermitian metric $d^A{}_B$ defined such that

$$C_{AB} \equiv \rho_{AC} d^C{}_B = -C_{BA}. \tag{4.87}$$

By redefinitions of the basis [56] one may diagonalize d while simultaneously bringing ρ to a canonical form

$$d = \begin{pmatrix} \mathbb{1}_{2p} & 0 \\ 0 & -\mathbb{1}_{2(r-p)} \end{pmatrix}, \qquad \rho = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{4.88}$$

These matrices should be covariantly constant. As we use a basis where they are actually constant, this implies

$$\mathcal{D}_X \rho_{AB} = -\omega_{XA}{}^C \rho_{CB} - \omega_{XB}{}^C \rho_{AC} = -\omega_{XAB} + \omega_{XBA}. \tag{4.89}$$

Thus the USp-connection is symmetric in such bases. There is even one further condition, which can be taken to be

$$d^A{}_C \omega_{XB}{}^C = \omega_{XC}{}^A d^C{}_B. \tag{4.90}$$

This is trivial in case $d = \mathbb{1}$, but not in general.

The subgroup of $\mathrm{G}\ell(r, \mathbb{H})$ that preserves d is $\mathrm{USp}(2p, 2r-2p)$. We define then the metric of the manifold to be

$$g_{XY} = f_X^{iA} C_{AB} \varepsilon_{ij} f_Y^{jB} = f_{XiA} d^A{}_B f_Y^{iB}. \tag{4.91}$$

Exercise 4.7: Check that for the simplest hypermultiplet with $n_H = 1$, one can take $\rho_{AB} = \varepsilon_{AB}$ and for $f_X{}^{iA}$ as non-zero elements

$$f_1^{12} = f_1^{21} = \mathrm{i}\frac{1}{\sqrt{2}}, \qquad -f_2^{12} = f_2^{21} = \frac{1}{\sqrt{2}},$$
$$f_3^{11} = -f_3^{22} = \mathrm{i}\frac{1}{\sqrt{2}}, \qquad f_4{}^{11} = f_4^{22} = \frac{1}{\sqrt{2}}. \tag{4.92}$$

This leads to $g_{XY} = \delta_{XY}$ and zero connections Γ^Z_{XY} and $\omega_{XA}{}^B$ and hence also zero $W_{ABC}{}^D$. Check also that the 2-form complex structures are, with $\alpha = 1, 2, 3$,

$$(J^\gamma)_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma}, \qquad (J^\gamma)_{\alpha 4} = -(J^\gamma)_{4\alpha} = \delta^\gamma_\alpha. \tag{4.93}$$

As we defined the raising and lowering of symplectic indices with the matrix ρ, and defined f_{iA}^{X} as the inverse of f_X^{iA} in (4.26), we have

$$f_{XiA} = f_X^{jB}\rho_{BA}\varepsilon_{ji} = f_X^{jB}C_{BC}\varepsilon_{ji}(d^{-1})^C{}_A = g_{XY}f_{iC}^{Y}(d^{-1})^C{}_A. \tag{4.94}$$

Therefore, on f the lowering of an index X with the metric is only valid when we compensate at the same time with the matrix d. E.g.

$$f_{iA}^{X}\, g_{XY} = f_{YiB}\, d^B{}_A\,, \qquad g^{XY} f_Y^{iA} = d^A{}_B f^{XiB}\,. \tag{4.95}$$

Gauging hypermultiplet isometries. Having introduced a metric, the symmetries should respect the metric. In other words, they should be isometries, which is expressed by the Killing equation

$$\mathfrak{D}_{(X}k_{Y)I} = 0\,. \tag{4.96}$$

As mentioned before, see (4.53), these have to be triholomorphic Killing vectors. Then, moment maps $\vec{P}_I$ can be defined by

$$\partial_X \vec{P}_I = -\tfrac{1}{2}\vec{J}_{XY}k_I^Y\,. \tag{4.97}$$

In rigid supersymmetry this allows for Fayet–Iliopoulos terms as integration constants in this equation. But when we impose conformal symmetry in view of coupling to supergravity, then the moment maps are determined to be

$$\vec{P}_I = -\tfrac{1}{6}k^X \vec{J}_X{}^Y k_I^Z g_{YZ}. \tag{4.98}$$

The preservation of the metric, encoded in the hermitian matrix $d^B{}_A$, imply on the transformation matrices $t_{IA}{}^B$, defined in (4.55), the restriction

$$d^C{}_B t_{IA}{}^B = t_I{}^C{}_B d^B{}_A\,. \tag{4.99}$$

This implies that the gauged isometries are contained in $\mathrm{USp}(2p, 2r-2p)$.

Remark on the conformal symmetry. Due to the fact that we have now a metric available, we can rewrite the homothetic Killing equation (4.44) and similar as in (2.43) introduce a scalar function such that

$$k_{DX} = g_{XY}k_D^Y = \partial_X k_D \tag{4.100}$$

This scalar function can then be used to generate the metric.

$$g_{XY} = \mathfrak{D}_X \partial_Y k_D. \tag{4.101}$$

To 4 dimensions. We can multiply (4.84) at both sides with a chiral projection P_R. Using the rules (4.63) we should now have for the action $S_{\rm hyper}$

$$\frac{\delta S_{\rm hyp}}{\delta\bar\zeta_A} = -2d^A{}_B\Gamma^B\,. \tag{4.102}$$

We also want the action to generate the field equations for the scalars that we have seen before. This leads to

$$\begin{aligned} S_{\rm hyp} \;=\; \int {\rm d}^4x \quad & \Big(-\tfrac12 g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y - \left(\bar\zeta_A\not{\mathcal{D}}\zeta^B d^A{}_B + \text{h.c.}\right) + \tfrac12 W_{BC}{}^{DA}d^E{}_A\bar\zeta_E\zeta_D\bar\zeta^B\zeta^C \\ & -g\left(2\bar X^I t_I{}^{BA}\bar\zeta_C\zeta_B d^C{}_A + 2{\rm i}f_X^{iA}k_I^X\bar\zeta_B\Omega^{jI}\varepsilon_{ij} + \text{h.c.}\right) - gP_{Ii}{}^kY^{ijI}\varepsilon_{jk} \\ & -2g^2\bar X^I X^J k_{(I}^X k_{J)}^Y g_{XY}\Big)\,. \end{aligned} \tag{4.103}$$

This satisfies (4.102) and also

$$\frac{\delta S_{\rm hyp}}{\delta q^X} = g_{XY}\Delta^Y + \left(2\bar\zeta_A\Gamma^B\omega_{XB}{}^C d^A{}_C + \text{h.c.}\right)\,. \tag{4.104}$$

The covariant derivatives in (4.103) read

$$\begin{aligned} \mathcal{D}_\mu q^X &= \partial_\mu q^X + gW_\mu^I k_I^X\,,\\ \mathcal{D}_\mu\zeta^A &= \partial_\mu\zeta^A + \partial_\mu q^X\omega_{XB}{}^A\zeta^B + \tfrac14\omega_\mu{}^{bc}\gamma_{bc}\zeta^A + gW_\mu^I t_{IB}{}^A\zeta^B\,. \end{aligned} \tag{4.105}$$

Exercise 4.8: Take our simple example, with vielbeins defined in (4.92). A triholomorphic Killing vector is e.g.

$$k^1 = q^2,\qquad k^2=-q^1,\qquad k^3=q^4,\qquad k^4=-q^3\,. \tag{4.106}$$

When the last two signs are interchanged, the vector is still a Killing vector, but is not triholomorphic.

The last one allows us to define a mass term. Indeed, consider the gauging by a trivial vector multiplet, where the scalar field X is fixed to a mass parameter m/g (to cancel the factor g^2 in the last line of (4.103)). Remark that this is still a vector multiplet, and if there are other vector multiplets, we take the prepotential F independent of this constant vector multiplet. In particular, also the corresponding field Y^{ij} in that vector multiplet vanishes.

In the simple one multiplet case, the matrix $t_{IA}{}^B$ is ${\rm i}(\sigma_3)_A{}^B$. The action (4.103) reduces to ($X=1,\ldots,4$ and $A=1,2$)

$$S_{\rm hyp} = \int {\rm d}^4x\left(-\tfrac12\partial_a q^X\partial^a q^X - 2\bar\zeta_A\not{\partial}\zeta^A + 4{\rm i}m\left(\bar\zeta_1\zeta_2 - \bar\zeta^1\zeta^2\right) - 2m^2q^Xq^X\right)\,. \tag{4.107}$$

Local superconformal result in 4 dimensions. After gauge covariantization and using the values of the conformal gauge fields as in (4.80), the final result of the action is in 4 dimensions

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm hyp,conf} &= -\tfrac12 g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y + \Big(-\bar\zeta_A\not{D}\zeta^B d^A{}_B + \text{h.c.}\Big) + \tfrac14 Dk_D^2 - \tfrac1{12}Rk_D^2\\
&\quad -\tfrac14\bar\psi^i_a\gamma^{abc}\psi_{bj}\mathcal{D}_c q^X J_X{}^Y{}_i{}^j k_{DY} + \tfrac12 W_{BC}{}^{DA}\bar\zeta_E\zeta_D\bar\zeta^B\zeta^C d^E{}_A\\
&\quad +\Big\{\tfrac12{\rm i}\bar\zeta_A\gamma^a\not{\mathcal{D}}q^X\psi_{ai}f^{iB}_X d^A{}_B + \tfrac1{12}\bar\psi^i_a\gamma^{abc}\mathcal{D}_b\psi_{ci}k_D^2 - \tfrac13{\rm i}A^{iB}\bar\zeta_A\gamma^{ab}\mathcal{D}_a\psi_{bi}d^A{}_B\\
&\quad +\tfrac18 k_D^2\bar\psi^i_a\gamma^a\chi_i - 2{\rm i}A^{iB}\bar\zeta_A\chi_i d^A{}_B + \tfrac1{12}{\rm i}A^{iB}\bar\zeta_A\gamma_a\psi^j_b T^{+ab}\varepsilon_{ij}d^A{}_B\\
&\quad -\tfrac1{48}k_D^2\bar\psi^i_a\psi^j_b\varepsilon_{ij}T^{+ab} + \tfrac18\bar\zeta_A\gamma\cdot T^+\zeta_B d^{AB} - g\Big({\rm i}\bar X^I k_I^X\bar\zeta_A\gamma^a\psi^j_a\varepsilon_{ij}f^{iB}_X d^A{}_B\\
&\quad +2\bar X^I t_I{}^{BA}\bar\zeta_C\zeta_B d^C{}_A + 2{\rm i}k_I^X f^{iA}_X\bar\zeta_B\Omega^{jI}\varepsilon_{ij} - \bar\psi_{aj}\gamma^a\Omega^I_i P_I^{ij} - \bar X^I\bar\psi^i_a\gamma^{ab}\psi^j_b P_{Iij}\Big)\\
&\quad -2g^2\bar X^I X^J k^X_{(I}k^Y_{J)} + \text{h.c.}\Big\} - gP_{Iij}Y^{ij}\,,
\end{aligned} \tag{4.108}$$

where we remember the definition (4.46), and the covariant derivatives are:

$$\begin{aligned}
\mathcal{D}_\mu q^X &\equiv \partial_\mu q^X - b_\mu k_D^X - V_{\mu i}{}^j k^X{}_j{}^i + gW_\mu^I k_I^X\,,\\
D_\mu\zeta^A &\equiv \mathcal{D}_\mu\zeta^A - \tfrac12{\rm i}f^{iA}_X\not{D}q^X\psi_{\mu i} - {\rm i}A^{iA}\phi_{\mu i} - {\rm i}g\bar X^I k_I^X f^{iA}_X\varepsilon_{ij}\psi^j_\mu\,,\\
\mathcal{D}_\mu\zeta^A &\equiv \partial_\mu\zeta^A + \partial_\mu q^X\omega_{XB}{}^A\zeta^B - \tfrac32 b_\mu\zeta^A - \tfrac12{\rm i}A_\mu\zeta^A + \tfrac14\omega_\mu{}^{ab}\gamma_{ab}\zeta^A\\
&\quad +gW_\mu^I t_{IB}{}^A\zeta^B\,.
\end{aligned} \tag{4.109}$$

5 Gauge fixing of superconformal symmetries

We have obtained the superconformal actions. We are now in the position to make the step to super-Poincaré symmetry. We thus have to break the superfluous symmetries.

A gauge symmetry means that there is a degree of freedom in the set of fields that is in fact absent from the action. Indeed, the gauge invariance can be written as the equation

$$\frac{\partial S}{\partial\phi^i}\delta\phi^i = 0\,, \tag{5.1}$$

where ϕ^i stands here for all the fields. If one can redefine the basis of fields such that all fields are inert apart from one field, then this equation says that this field does not occur in the action. In principle this can always be done, but the field redefinitions are often non-local. If there is one field that has the same number of degrees of freedom than the gauge symmetry itself, one can often make such redefinitions. However, this procedure is often cumbersome while the result is just the same as taking a 'gauge choice' for this symmetry. The example presented in section 2.3 can illustrate these remarks.

In this section we want to consider the geometries that result after the gauge fixings. But before starting on the geometry, let us first discuss the gauge fixing of special conformal transformations. A natural gauge choice for K transformations is

$$K\text{-gauge:}\quad b_\mu = 0\,. \tag{5.2}$$

Note that if one imposes such a gauge condition, it does not imply that one should forget about K transformations. The correct conclusion is that now the K transformations are dependent on the other ones, in such a way that the gauge condition is respected. Though the principles that we discuss here are general, *we will concentrate in this chapter on $d = 4$ dimensions.*

In this case, comparing (3.10) and (3.70), we obtain

$$\Lambda_K^a = -\tfrac{1}{2}e^{\mu a}\left[\partial_\mu\Lambda_D + \tfrac{1}{2}\left(\bar\epsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \bar\eta^i\psi_{\mu i} + \text{h.c.}\right)\right] . \tag{5.3}$$

Such a rule is called a *decomposition law.*

E.g. the first term implies that what one denoted above as special conformal transformations, is a contribution to local dilatations of the action where b_μ is omitted.

XXX Here example pure $N = 2$, $d = 4$.

5.1 Projective Kähler geometry

We now first concentrate on the bosonic part of the action, and in particular on the part related to the scalars of the vector multiplets. This part defines special Kähler geometry, but we will first be present it in a more general way so that it even applies to $N = 1$ supergravity and explains the concept of a projective Kähler geometry. Let us write the relevant terms of the Lagrangian as

$$e^{-1}\mathcal{L}_N = -g^{\mu\nu}N_{I\bar J}(X,\bar X)\partial_\mu X^I\partial_\nu\bar X^{\bar J} + \tfrac{1}{2}aN(X,\bar X)R\,, \qquad N(X,\bar X) \equiv -N_{I\bar J}(X,\bar X)X^I\bar X^{\bar J} \tag{5.4}$$

This is similar to the result in (4.81), but with some modifications. First, the matrix $N_{I\bar J}$ is not necessary the N_{IJ} of (4.82), but satisfies as the latter the equation

$$N_{I\bar J} = -\frac{\partial}{\partial X^I}\frac{\partial}{\partial\bar X^{\bar J}}N\,. \tag{5.5}$$

This equation implies first of all that $N_{I\bar J}$ is a Kähler metric, with Kähler potential $-N$, and second that X^I is a closed homothetic Killing vector of this metric, see (2.30) with $w = 1$, which is satisfied due to the consequences of the previous equation:

$$\frac{\partial}{\partial X^K}N_{I\bar J} = \frac{\partial}{\partial X^I}N_{K\bar J}\,, \qquad X^K\frac{\partial}{\partial X^K}N_{I\bar J} = 0\,, \tag{5.6}$$

and their complex conjugates. For $N = 2$ vector multiplets, these homogeneity properties follow from (4.70).

Therefore there can be a conformal symmetry. In fact, (5.4) would be local conformal invariant if $a = 1/3$. This also agrees with (4.81). However, that equation has also a term DN. We will come back to this later, but one can already easily check that eliminating the auxiliary field D from the sum of (4.81) and (4.108) modifies the coefficient a in (5.4) to $a = 1$. Therefore we keep a arbitrary, the value $a = 1/3$ corresponds to what one finds for $N = 1$ supergravity, while we have $a = 1$ in $N = 2$ supergravity. We will thus continue here with (5.4) and show how a 'projective Kähler manifold' emerges from this Kähler manifold with the closed homothetic Killing vector. A more extensive discussion of this construction appears in [59].

The presence of a complex structure and a closed homothetic Killing vector implies in general a U(1) symmetry. Here, this is the phase transformation of the vector X^I, and we know this U(1) transformation already from its presence in the superconformal algebra. For the fields of the vector multiplet, see table 4, i.e.

$$\delta_{\mathrm{U}(1)} X^I = -\mathrm{i}\Lambda_A X^I\,, \qquad \delta_{\mathrm{U}(1)} \bar X^{\bar I} = \mathrm{i}\Lambda_A \bar X^{\bar I}\,. \tag{5.7}$$

In order to promote this symmetry to a gauge symmetry, it should be gauged by a gauge field, which is the field A_μ that we encountered in the Weyl multiplet, see e.g. table 3. Therefore we have to replace the derivatives in (5.4) by covariant derivatives

$$\partial_\mu X^I \to \mathcal{D}_\mu X^I = \left(\partial_\mu + \mathrm{i}A_\mu\right) X^I\,, \tag{5.8}$$

which is what we had in (4.83), where we have already put (5.2), and we take for now the Abelian theory (or neglect the gauge terms proportional to g).

The field equation for the auxiliary field A_μ is algebraic and it allows us to solve for A_μ giving it the value

$$\tilde A_\mu = \frac{\mathrm{i}}{2N}\left[X^I N_{I\bar J}(\partial_\mu \bar X^{\bar J}) - (\partial_\mu X^I) N_{I\bar J}\bar X^{\bar J}\right] = -\frac{\mathrm{i}}{2N}\left(\partial_\mu X^{\bar J}\partial_{\bar J}N - \partial_\mu X^I \partial_I N\right)\,. \tag{5.9}$$

The first term of (5.4) is then

$$\begin{aligned}
\frac{\mathcal{L}_{\mathrm{scalar}}}{\sqrt{g}} &= -g^{\mu\nu} N_{I\bar J}(X,\bar X)\mathcal{D}_\mu X^I \mathcal{D}_\nu \bar X^{\bar J}\\
&= -N_{I\bar J}\partial_\mu X^I \partial^\mu \bar X^{\bar J} + \frac{1}{4N}\left[\partial_\mu \bar X^{\bar J}\partial_{\bar J}N - \partial_\mu X^I\partial_I N\right]^2\\
&= \partial_\mu X^I \partial^\mu \bar X^{\bar J}\left[\partial_I\partial_{\bar J}N - \frac{1}{N}(\partial_I N)(\partial_{\bar J}N)\right] + \frac{1}{4N}\left[\partial_\mu N\,\partial^\mu N\right]\\
&= N\partial_\mu X^I \partial^\mu \bar X^{\bar J}\partial_I \bar\partial_{\bar J}\log|N| + \frac{1}{4N}\left[\partial_\mu N\,\partial^\mu N\right]\,.
\end{aligned} \tag{5.10}$$

As in the example in section 2.3, we take the gauge fixing such that the second term of (5.4) reduces to the Einstein-Hilbert action. Hence, we take

$$D\text{-gauge:}\quad N = \frac{1}{a\kappa^2}\,. \tag{5.11}$$

The resulting scalar manifold is thus the submanifold of the scalars defined by this condition. It will be parametrized by n complex fields z^α (and complex conjugates $\bar z^{\bar\alpha}$). We define

$$X^I = Y Z^I(z)\,, \qquad \bar X^I = \bar Y \bar Z^I(\bar z)\,, \tag{5.12}$$

where $Z^I(z)$ are $n+1$ non-degenerate[20] arbitrary holomorphic functions of the z^α, and Y is the $(n+1)$th complex variable. The latter will be determined by the condition (5.11) and by a gauge choice for U(1) in terms of the Z, $\bar{Z}$ variables[21], such that we get functions $Y(Z,\bar{Z})$.

Notice that the transition from X^I to Z^I does not respect the holomorphicity. In other words, the complex structure that is relevant in the submanifold is not the same as that in the embedding manifold. If we denote by Φ_Y the map in the embedding manifold $\tilde{\mathcal{M}}$

$$\begin{aligned}\Phi_Y : \tilde{\mathcal{M}} &\longrightarrow \tilde{\mathcal{M}}\\ Z^I &\longrightarrow X^I = Z^I Y(Z,\bar{Z})\,,\end{aligned}\qquad(5.13)$$

then the old complex structure J on $\tilde{\mathcal{M}}$ induces a new complex structure, denoted as J', by the commutativity of the diagram

$$\begin{array}{ccc} T\tilde{\mathcal{M}} & \xrightarrow{T\Phi_Y} & T\tilde{\mathcal{M}} \\ {\scriptstyle J'}\big\downarrow & & \big\downarrow{\scriptstyle J} \\ T\tilde{\mathcal{M}} & \xrightarrow[T\Phi_Y]{} & T\tilde{\mathcal{M}} \end{array}$$

The map $T\Phi_Y$ then sends holomorphic vectors with respect to J' to holomorphic vectors with respect to J. In this sense, it is a holomorphic map.

As N gets a fixed value, a function of N is not convenient as a Kähler potential for the restricted manifold. We will show now how to construct a Kähler potential.

Defining

$$\mathcal{K}(z,\bar{z}) = -\frac{1}{a\kappa^2}\ln\left[a\frac{N}{Y\bar{Y}}\right] = -\frac{1}{a\kappa^2}\ln\left[-aZ^I(z)N_{I\bar{J}}(z,\bar{z})\bar{Z}^{\bar{J}}(\bar{z})\right]\,,\qquad(5.14)$$

we can use the homogeneity properties to reduce the action on the surface (5.11) to

$$e^{-1}\mathcal{L}_N = -\partial_\mu z^\alpha\partial^\mu\bar{z}^{\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\mathcal{K} + \frac{1}{2\kappa^2}R\,.\qquad(5.15)$$

Proof.

With the coordinates as in (5.12), the last part of (5.6) implies that $N_{I\bar{J}}$ depends only on z and $\bar{z}$, and not on Y. The condition (5.6) implies that when we derive $Z^I N_{I\bar{J}}\bar{Z}^{\bar{J}}$ with respect to z we can forget derivatives on $N_{I\bar{J}}\bar{Z}^{\bar{J}}$, and hence

$$\begin{aligned} a\kappa^2\partial_\alpha\mathcal{K} &= -\frac{N_{I\bar{J}}\bar{Z}^{\bar{J}}\partial_\alpha Z^I}{N_{K\bar{L}}Z^K\bar{Z}^{\bar{L}}}\,,\\ a\kappa^2\partial_\alpha\partial_{\bar{\beta}}\mathcal{K} &= \partial_\alpha Z^I\partial_{\bar{\beta}}\bar{Z}^{\bar{J}}Y\bar{Y}\left[\frac{N_{IJ}}{N} + \frac{N_{I\bar{K}}\bar{X}^{\bar{K}}X^L N_{L\bar{J}}}{N^2}\right]\\ &= -\partial_\alpha Z^I\partial_{\bar{\beta}}\bar{Z}^{\bar{J}}Y\bar{Y}\partial_I\partial_{\bar{J}}\ln N\,.\end{aligned}\qquad(5.16)$$

[20]The matrix $\partial_\alpha Z^I$ has to be of rank n and the matrix $(Z^I,\partial_\alpha Z^I)$ has to be of rank $n+1$.

[21]Note that in practice, we will not need to specify a U(1) gauge choice in this derivation. The phase of Y disappears automatically because it is the only quantity that transforms under U(1).

Using $\tilde{\mathcal{D}}_\mu$ for (5.8) with A replaced by $\tilde{A}$, we have

$$\tilde{\mathcal{D}}_\mu X^I = Y\,\partial_\mu z^\alpha\,\mathcal{D}_\alpha Z^I + \tfrac{1}{2}X^I\partial_\mu \ln N\,,\qquad \mathcal{D}_\alpha Z^I \equiv \left[\partial_\alpha + (\partial_\alpha\mathcal{K})\right] Z^I\,. \tag{5.17}$$

Notice that $\tilde{\mathcal{D}}_\mu X^I N_{I\bar{J}}\bar{X}^{\bar{J}}$ is real, which eliminates the linear terms in $A_\mu - \tilde{A}_\mu$ in the Lagrangian.

Plugging this in the Lagrangian (and for future use re-installing the term with the auxiliary field) one obtains

$$e^{-1}\mathcal{L}_{\rm scalar} = \tfrac{1}{4}N^{-1}\left(\partial_\mu N\right)^2 + N\left(A_\mu - \tilde{A}_\mu\right)^2 - Y\bar{Y}N_{IJ}\mathcal{D}_{\bar{\alpha}}\bar{Z}^I\,\mathcal{D}_\beta Z^J\,\partial_\mu\bar{z}^{\bar{\alpha}}\,\partial^\mu z^\beta\,. \tag{5.18}$$

Thus, before any gauge fixing we can write the action as

$$e^{-1}\mathcal{L}_{\rm scalar} = \tfrac{1}{4}N^{-1}\left(\partial_\mu N\right)^2 + N\left(A_\mu - \tilde{A}_\mu\right)^2 - N\left(\partial_{\bar{\beta}}\partial_\alpha\mathcal{K}\right)\partial_\mu z^\alpha\,\partial^\mu z^{\bar{\beta}}\,, \tag{5.19}$$

where $\tilde{A}_\mu$ is the value of A_μ that we wrote in (5.9), and which for future use we write also in terms of the z variables:

$$\tilde{A}_\mu = -\tfrac{1}{2}{\rm i}a\kappa^2\left(\partial_\alpha\mathcal{K}\,\partial_\mu z^\alpha - \partial_{\bar{\alpha}}\mathcal{K}\,\partial_\mu\bar{z}^{\bar{\alpha}}\right) + \tfrac{1}{2}{\rm i}\partial_\mu\ln\frac{Y}{\bar{Y}}\,. \tag{5.20}$$

This proves (5.15) after N is put equal to a constant and use of the field equation for A_μ.

Comments

Note that the normalization of the Kähler potential (5.14) depends on the value a, which we have discussed in the beginning of this section. This leads to the familiar factors 3 that appear in these formulae for $N = 1$.

When we now restrict to $N = 2$,

$$N_{I\bar{J}} \;\rightarrow\; N_{IJ}\,,\qquad a = 1\,. \tag{5.21}$$

where N_{IJ} was defined in (4.82). The gauge fixing for dilatations is the condition

$$D\text{-gauge:}\quad N = -X^I N_{IJ}X^J = {\rm i}(\bar{X}^I F_I - \bar{F}_I X^I) = |Y|^2{\rm e}^{-\kappa^2\mathcal{K}} = \kappa^{-2}\,. \tag{5.22}$$

In the full theory, (5.22) is not invariant under all gauge transformations, e.g. under supersymmetry. But to simplify the case here, the S-gauge condition is taken as the Q-supersymmetry variation of this gauge, such that the 'decomposition law' is just $\Lambda_D = 0$. Using the homogeneity properties of the functions F, (4.70), it is easy to see that the transformation under a chiral supersymmetry of (5.22) leads to

$$S\text{-gauge:}\quad X^I N_{IJ}\Omega^{iJ} = 0\,. \tag{5.23}$$

This implies another decomposition law expressing η in terms of a field-dependent ϵ. The resulting combination of the original Q-supersymmetry transformations and those induced by S-supersymmetry is the Poincaré supersymmetry.

Positivity.

Note that there are positivity conditions restricting the domain of the scalars, and the form of the matrix N. First of all in order that the gauge condition (5.28) can be satisfied,

$$Z^I N_{IJ} \bar{Z}^J < 0\,, \tag{5.24}$$

and thus $N_{IJ}(z,\bar{z})$ should have at least one negative eigenvalue for all values of the scalars in the domain.

On the other hand, to have positive kinetic energy of the physical scalars, one has to impose positivity of (5.16). For this one needs the non-triviality condition that the matrix $\partial_\alpha X^I$ has to be of rank n.

Remaining problems: fixing SU(2) and a field equation.

I have neglected the fact that there is still the local SU(2). I could do so because the scalars of the vector multiplets are invariant under SU(2). But for the full super-Poincaré gravity, I have to take that into account. A second problem is the field equation of the field D that sits in the Weyl multiplet. If I would have written the full action that I have constructed so far, it would contain a term

$$\int \mathrm{d}^4x\, eD\, N\,. \tag{5.25}$$

where D is that field of the Weyl multiplet. Its field equation would thus imply $N = 0$, and I cannot impose the gauge that I was proposing.

Both problems are solved by introducing another 'compensating' multiplet involving scalars that do transform under SU(2). These can then be used to fix the SU(2). In the context of what has been introduced here, the obvious candidate is a hypermultiplet. But other possibilities exist as well. I will explain how this solves also the problem of the inconsistent field equation after the hypermultiplet action has been introduced. For following further the aspects of vector multiplet couplings only, you may just neglect the term (5.25), as will be explained later.

The scheme that extends the scheme (4.5) is thus:

$$\begin{array}{c}
\text{Weyl multiplet: } e_\mu{}^a,\ b_\mu,\ \psi_\mu^i,\ V_{\mu i}{}^j,\ A_\mu,\ T_{ab},\ \chi^i,\ D \\
+ \\
\text{vector multiplet: } z,\ \Omega^i,\ W_\mu,\ Y_{ij} \\
+ \\
\text{second compensating multiplet: } A_i^\alpha,\ \zeta^\alpha,\ F_i^\alpha \\
\qquad\qquad\downarrow \text{ gauge fixing } K_a\, D,\ U(1),\ S^i,\ SU(2) \\
N=2 \text{ super-Poincaré gravity } e_\mu{}^a\ \psi_\mu^i,\ W_\mu,\ V_{\mu i}{}^j,\ A_\mu,\ T_{ab},\ \chi^i,\ D,\ Y_{ij},\ F_i^\alpha\,.
\end{array} \tag{5.26}$$

In this scheme appears thus just the one vector multiplet that leads only to the graviphoton and includes no other propagating fields. For the second compensating multiplet, I wrote the scheme with as second compensating multiplet a hypermultiplet, including auxiliary fields

F_i^α. (We will not further discuss auxiliary fields for the hypermultiplets in these lectures). In the remaining set of fields of the $N = 2$ super-Poincaré theory, the first ones are the physical graviton, the gravitinos and the graviphoton W_μ. All the others are auxiliary fields, and the full set includes 40+40 off-shell degrees of freedom. There are two alternative choices known for the second compensating multiplet. These multiplets are the so-called 'nonlinear multiplet' and the 'linear multiplet'. In each case one gets a different set of auxiliary fields for the $N = 2$ super-Poincaré theory, with the same number of field components.

One denotes by gauged $N = 2$ supergravity the situation where the gravitinos transform under a local $SU(2)$, and where this induces then also a cosmological constant (anti-de Sitter supergravity). This is obtained in the present formalism if the scalar compensating multiplet transforms under the $U(1)$ gauge group that is gauged by the first compensating multiplet. Also the full automorphism group $SU(2)$ can be gauged by having the second compensating multiplet transforming under an $SU(2)$ group. In both situations, the gauge fixing on the scalars of this multiplet then mixes the $SU(2)$ factor of the superconformal group with the group gauged by vector multiplets. The first compensating multiplet can even be part of a non-compact $SO(2,1)$ gauge group [60] leading to a massive vector multiplet.

5.2 Interpretation as Sasakian cone

The obtained metric is a cone [61, 51]. We see this as follows. We have split the $n + 1$ complex variables $\{X\}$ in $\{Y, z^\alpha\}$, and can then split Y in $Y = |Y|e^{\mathrm{i}\theta}$. The modulus appears in N, which we can use as a real coordinate. To obtain a canonical parametrization of a cone we define $r^2 = N$. We thus have

- r is scale which is a gauge degree of freedom for translations
- θ is the U(1) degree of freedom;
- the n complex variables z^α.

The metric (5.19) now takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \left[A + d\theta + \tfrac{1}{2}\mathrm{i}\left(\partial_\alpha\mathcal{K}(z,\bar z)\, dz^\alpha - \partial_{\bar\alpha}\mathcal{K}(z,\bar z)\, d\bar z^{\bar\alpha}\right)\right]^2 - r^2\partial_\alpha\partial_{\bar\alpha}\mathcal{K}(z,\bar z)\, dz^\alpha d\bar z^{\bar\alpha}\,, \tag{5.27}$$

where A is the one-form gauging the U(1) group, and $\mathcal{K}(z,\bar z)$ is a function of the holomorphic prepotential $F(X)$. When U(1) is not gauged ($A_\mu = 0$), the base of the cone (the manifold with fixed N) is a Sasakian manifold with a U(1) invariance[22]. Here, in supergravity, we gauge U(1), which implies that the auxiliary field A_μ can be redefined such that the whole expression in square brackets is A, and it drops out by its field equation. With fixed r (gauge fixing the superfluous dilatations), the remaining manifold is Kähler, with the Kähler potential $\mathcal{K}$ determined by $F(X)$. That gives the special Kähler metric. The Kähler manifold is thus the submanifold of the $(n + 1)$-complex-dimensional manifold defined by a constant

[22]This has been remarked first in a similar situation with hypermultiplets in $N = 2$ in [51], and has been looked at systematically in [61].

value of r or N. This is a real condition, but the U(1) invariance implies that the other real variable θ has disappeared.

We will adopt the dilatational gauge-fixing condition

$$D\text{-gauge:}\quad N=1, \tag{5.28}$$

which thus can be written as

$$|Y|^2=\mathrm{e}^{\mathcal{K}}. \tag{5.29}$$

One can conveniently chose the U(1) gauge

$$U(1)\text{-gauge:}\quad Y=\bar{Y} \tag{5.30}$$

leading to

$$Y=\mathrm{e}^{\mathcal{K}/2}. \tag{5.31}$$

5.3 The resulting bosonic action and examples

I recapitulate now what we have found. First, I write a general formulation for the bosonic sector of a theory with scalar fields z^α, and vector fields labelled by an index I. If there are no Chern–Simons terms (these do occur for non-Abelian theories if $\delta F\neq 0$), one can write a general expression

$$\begin{aligned}\mathcal{L}_0 &= -e\,g_{\alpha\bar\beta}D_\mu z^\alpha D^\mu\bar z^\beta\ -\ V(z) \qquad (5.32)\\ \mathcal{L}_1 &= \tfrac12 e\ \mathrm{Im}\left(\mathcal{N}_{IJ}(z)\,F^{+I}_{\mu\nu}F^{+\mu\nu\,J}\right)=\tfrac14 e(\mathrm{Im}\,\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J}-\tfrac18(\mathrm{Re}\,\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\,.\end{aligned}$$

$g_{\alpha\bar\beta}$ is the (positive definite) metric of the target space, while $\mathrm{Im}\,\mathcal{N}$ is a (negative definite) matrix of the scalar fields, whose vacuum expectation value gives the gauge coupling constants, while that of $\mathrm{Re}\,\mathcal{N}$ gives the so-called theta angles. V is the potential.

The embedding of this hypersurface can be described in terms of n complex coordinates z^α by letting X^I be proportional to some holomorphic sections $Z^I(z)$ of the projective space $\mathbb{CP}^n$ [62, 63, 64]. The n-dimensional space parametrized by the z^α ($\alpha=1,\ldots,n$) is a Kähler space; the Kähler metric $g_{\alpha\bar\beta}=\partial_\alpha\partial_{\bar\beta}\mathcal{K}(z,\bar z)$ follows from the Kähler potential

$$\begin{aligned}&\mathcal{K}(z,\bar z)=-\ln\left[\mathrm{i}\bar Z^I(\bar z)\,F_I(Z(z))-\mathrm{i}Z^I(z)\,\bar F_I(\bar Z(\bar z))\right]\,,\qquad\text{where} \qquad (5.33)\\ &X^I=e^{\mathcal{K}/2}Z^I(z)\,,\qquad \bar X^I=e^{\mathcal{K}/2}\bar Z^I(\bar z)\,.\end{aligned}$$

The resulting geometry is known as *special* Kähler geometry [65, 1, 66].

I have not yet commented on Kähler transformations. Their origin sits in the split (5.12). At that point this splitting is not unique and correspondingly there are transformations

$$Y'=Y\,\mathrm{e}^{\Lambda_K(Z)},\qquad Z'^I=Z^I\,\mathrm{e}^{-\Lambda_K(Z)}. \tag{5.34}$$

Under the gauge condition $N=1$ they imply, by (5.14) a transformation

$$\mathcal{K}'(z,\bar z)=\mathcal{K}(z,\bar z)+\Lambda_K(z)+\bar\Lambda_K(\bar z). \tag{5.35}$$

If we take a U(1) gauge $Y = \bar{Y}$, then this leaves a combination of the U(1) and the Kähler transformation (5.34):

$$2\mathrm{i}\Lambda_A = \Lambda_K(Z) - \bar{\Lambda}_K(\bar{Z}) \tag{5.36}$$

as decomposition law. The remaining Kähler transformation can e.g. be used to choose one of the Z^I, say Z^0, equal to 1. In any case, one can choose the parametrization of the n physical scalars z^α (with $\alpha = 1, \ldots, n$) at random, as stressed in [62, 63, 64].

A convenient choice of inhomogeneous coordinates z^α are the *special* coordinates, defined by

$$z^\alpha = \frac{X^\alpha}{X^0}\,, \qquad \text{or} \qquad Z^0(z) = 1\,, \qquad Z^A(z) = z^\alpha\,. \tag{5.37}$$

In the general form of the spin-1 action (5.32), the matrix $\mathcal{N}$ is given by

$$\mathcal{N}_{IJ}(z,\bar{z}) = \bar{F}_{IJ} + \mathrm{i}\frac{N_{IN}N_{JK}\,X^N X^K}{N_{LM}\;X^L X^M}\,. \tag{5.38}$$

Exercise 5.1: Check the following useful relation:

$$\operatorname{Im}\mathcal{N}_{IJ}\bar{X}^J = -\frac{1}{2}\frac{N_{IJ}X^J}{N_{LM}\;X^L X^M}. \tag{5.39}$$

The kinetic energy terms of the scalars are positive definite in a so-called 'positivity domain'. This is given by the conditions $g_{\alpha\bar{\beta}} > 0$ and $e^{-\mathcal{K}} > 0$. Then it follows that $\operatorname{Im}\mathcal{N}_{IJ} < 0$ [67]. Note that the fact that the positivity domain is non-empty, restricts the functions F that can be used.

There is one global aspect [68, 69] which is important, and as far as I know it is the only instant where the fermion sector comes in. As we have seen, the fermions transform under the superconformal U(1) factor, and hence, by (5.36), under the (finite) Kähler transformations:

$$\Omega_i \to e^{-\frac{1}{4}\left(\Lambda_K(Z)-\bar{\Lambda}_K(\bar{Z})\right)}\Omega_i\,. \tag{5.40}$$

Then one argues in the same way as for the magnetic monopole (as nicely explained in [68, 69]): the fermion should remain well defined when going e.g. around a sphere. If fields transform as $\psi \to U\psi$, then the gauge field A_μ, normalized so that it transforms as $\partial_\mu + A_\mu \to U^{-1}(\partial_\mu + A_\mu)U$, has a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, of which the integral over an arbitrary 2-cycle should be

$$[F] \equiv \int F_{\mu\nu}\, dx^\mu\, dx^\nu = 2\pi \mathrm{i} n\;; \qquad n \in \mathbb{Z}\,. \tag{5.41}$$

The properly normalized gauge field is then

$$A_\alpha = -\tfrac{1}{4}\partial_\alpha K\;; \qquad A_{\bar{\alpha}} = \tfrac{1}{4}\partial_{\bar{\alpha}} K\,, \tag{5.42}$$

and the curvature,

$$F_{\alpha\bar{\beta}} = -F_{\bar{\beta}\alpha} = \tfrac{1}{2}g_{\alpha\bar{\beta}}\,, \qquad \mathcal{K} = \frac{\mathrm{i}}{2\pi}g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}}\,. \tag{5.43}$$

is proportional to the Kähler 2-form. We thus find

$$c_1 = \tfrac{1}{2}[\mathcal{K}] \in \mathbb{Z}\,. \tag{5.44}$$

We find that the Kähler form should be of even integer cohomology. If there had been no fermions present in the theory, the same argument applied just to the bosonic fields would have allowed an arbitrary integer cohomology. In the mathematical literature Kähler manifolds of which the Kähler form is of integer cohomology are called *Hodge manifolds*. With a slight abuse of language we will call *Hodge–Kähler manifold* a Kähler manifold with Kähler form of even integer cohomology.

We give here some examples of functions $F(X)$ and their corresponding target spaces:

$$F = -\mathrm{i}\, X^0 X^1 \qquad \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \tag{5.45}$$

$$F = (X^1)^3/X^0 \qquad \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \tag{5.46}$$

$$F = -4\sqrt{X^0(X^1)^3} \qquad \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \tag{5.47}$$

$$F = \tfrac{1}{4}\mathrm{i}X^I\eta_{IJ}X^J \qquad \frac{\mathrm{SU}(1,n)}{\mathrm{SU}(n)\otimes \mathrm{U}(1)} \tag{5.48}$$

$$F = \frac{d_{ABC}X^AX^BX^C}{X^0} \qquad \text{`very special'} \tag{5.49}$$

The first three functions give rise to the manifold $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$. However, the first one is not equivalent to the other two as the manifolds have a different value of the curvature [70]. The latter two are, however, equivalent by means of a symplectic transformation, as I will show in section 6.2. In the fourth example η is a constant non-degenerate real symmetric matrix. In order that the manifold has a non-empty positivity domain, the signature of this matrix should be $(- + \cdots +)$. The last example, defined by a real symmetric tensor d_{ABC}, with $A, B, C = 1, \ldots n$, defines a class of special Kähler manifolds, which are denoted as 'very special' Kähler manifolds. The corresponding supergravity theories are those that can be obtained from dimensional reduction of $d = 5$ supergravity–vector multiplet couplings.

Exercise 5.2: We go through the example (5.48) in some detail. We get easily the second derivative

$$F_{IJ} = \tfrac{1}{2}\mathrm{i}\eta_{IJ}\,, \qquad N_{IJ} = \eta_{IJ}\,. \tag{5.50}$$

We will now specify to

$$\eta_{IJ} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{5.51}$$

We thus write the formulae for $n = 1$, but they can be easily generalized for arbitrary n by taking the lower-right entry to be a unit $n \times n$ matrix. From (5.33) we get

$$\mathrm{e}^{-\mathcal{K}(z,\bar z)} = Z^0(z)\bar Z^0(\bar z) - Z^1(z)\bar Z^1(\bar z)\,. \tag{5.52}$$

The right-hand side should be positive. A convenient parametrization is thus $Z^0 = 1$, $Z^1 = z$. Indeed, then $\partial_z Z^I$ is of rank 1, while the 2×2 matrix (Z^I, ∂_z^I) is of rank 2. The domain for z is then $|z|^2 < 1$.

This leads to

$$\partial_z \mathcal{K} = \frac{\bar{z}}{1 - z\bar{z}}\,, \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} \mathcal{K} = \frac{1}{1 - z\bar{z}}\,, \tag{5.53}$$

You may check that the indefinite signature of (5.51) was necessary to have a positive metric.

The kinetic terms of the two vectors are determined by the matrix (5.38)

$$\mathcal{N}_{IJ} = \frac{1}{2}\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\mathrm{i}}{z^2 - 1}\begin{pmatrix} 1 & -z \\ -z & z^2 \end{pmatrix} = -\frac{\mathrm{i}}{1 - z^2}\begin{pmatrix} \frac{1}{2}(1 + z^2) & -z \\ -z & \frac{1}{2}(1 + z^2) \end{pmatrix}. \tag{5.54}$$

The imaginary part is negative definite as it should be for positive kinetic terms of the vectors.

5.4 Hypermultiplet action

The construction of special Kähler geometry that we have outlined here can be used as well for the quaternionic spaces (and for the real ones). We have constructed earlier the action of a hypermultiplet coupled to superconformal $N = 2$ Weyl multiplet (and possibly to vector multiplets). We use $r = n_H + 1$ hypermultiplets, and will decouple one quaternion (4 real fields) from the others. One real field will be a scale degree of freedom[23], three are SU(2) degrees of freedom, and the remaining ones form n_H quaternions. As in the metric of the vector multiplets, there is a connection to Sasakian manifolds. The $4(h_H + 1)$ dimensional manifold is a cone. The cone has a radial parameter and the dilatations scale from one to another value on the cone. Fixing that scale leads to a 3-Sasakian manifolds (of dimensions $4n_H + 3$), if one puts the gauge fields of the SU(2) invariance to zero, rather than using their field equations. Using their field equations, and eliminating the three gauge degrees of freedom, leads to the quaternionic manifold of dimension $4r$.

There is a result of Swann [71] that implies that every quaternionic manifold can be obtained in this way. The condition that a hyper-Kähler manifold can be formulated in a conformal way is equivalent to the condition that there is a cone. Therefore, those hyper-Kähler that satisfy this condition are one-to-one related with the hyper-Kähler manifolds that can be made quaternionic by an SU(2) gauging. The result of Swann has been made explicit by the construction that we will now explain [72]. The procedure is the same whether applied for $d = 4$, $d = 5$ or $d = 6$.

We will now use the notation $\hat{X}$ to denote the $4r = 4(n_H + 1)$ scalar fields in all the hypermultiplets. The notation X will then be reserved for the $4n_H$ scalars that remain after the gauge fixing. First, we choose in the hyper-Kähler manifold one coordinate, denoted

[23] When vector multiplets and hypermultiplets are simultaneously coupled, there is one overall dilatational gauge degree of freedom. An auxiliary field of the superconformal gauge multiplet gives a second relation, such that as well the compensating field of the vector multiplet as that of the hypermultiplet are fixed.

by w^0 in the direction of the dilatation vector (homothetic Killing vector) $\hat{k}_D^{\hat{X}}$. Further we choose 3 coordinates according to the SU(2) vector field (4.45). Our new basis is therefore of the form

$$\{q^{\hat{X}}\} = \{w^0, w^\alpha, q^X\}. \tag{5.55}$$

This choice is thus taken such that

$$\hat{k}_D^{\hat{X}}(w,q) = \{(d-2)w^0, 0, 0\}\,, \qquad \hat{\vec{k}}^{\hat{X}}(w,q) = \{0, \vec{k}^\alpha, 0\}\,. \tag{5.56}$$

Note that we use the index α here for the choice of coordinates and the vector sign for the 3 directions of the SU(2) vectors.

Similarly, the coordinates on the target space are now generically denoted by $\hat{A}$, and these can be split in $2+2n_H$. Mathematically, this can be done because the distinction of w^0 and w^α splits the structure group $\mathrm{G}\ell(n_H+1,\mathbb{H})$ to $\mathrm{SU}(2)\times\mathrm{G}\ell(n_H,\mathbb{H})$. We can thus write

$$\{\hat{A}\} = \{i, A\}. \tag{5.57}$$

Here, i is an SU(2)-index taking values 1 and 2. We thus have for the fermions

$$\{\zeta^{\hat{A}}\} = \{\zeta^i, \zeta^A\}. \tag{5.58}$$

In our approach this splitting is done such that only the ζ^i transform under S-supersymmetry.

The constraints of dilatation invariance and general properties of the quaternionic vielbeins and metric imply that the metric takes the form [72]

$$\begin{aligned} \mathrm{d}\hat{s}^2 &\equiv \hat{g}_{\hat{X}\hat{Y}}\mathrm{d}q^{\hat{X}}\mathrm{d}q^{\hat{Y}} \\ &= -\frac{(\mathrm{d}w^0)^2}{w^0} + w^0\left\{h_{XY}(q)\mathrm{d}q^X\mathrm{d}q^Y\right. \\ &\quad \left. -g_{\alpha\beta}(w,q)[\mathrm{d}w^\alpha + A_X^\alpha(w,q)\mathrm{d}q^X][\mathrm{d}w^\alpha + A_X^\alpha(w,q)\mathrm{d}q^X]\right\}, \end{aligned} \tag{5.59}$$

were we used the notations

$$A_X^\alpha(w,q) \equiv \hat{f}_{ij}^\alpha \hat{f}_X^{ij} = -\hat{f}_{iA}^\alpha \hat{f}_X^{iA}\,, \qquad \hat{g}_{\alpha\beta}(w,q) \equiv w^0 g_{\alpha\beta}\,. \tag{5.60}$$

After the gauge fixing, the final metric on the $4n_H$-dimensional space is $w^0 h_{XY}$. The metric $g_{\alpha\beta}$ is invertible in the 3-dimensional space of the w^α, and is used there to raise and lower α indices.

Quaternionic geometry. The results of [72] imply that for all values of w^0

$$g_{XY}(w,q) \equiv w^0 h_{XY}(q) = \hat{g}_{XY} + g_{\alpha\beta}A_X^\alpha A_Y^\beta \tag{5.61}$$

defines a quaternionic-Kähler metric. It has also been proven that any quaternionic-Kähler metric can be obtained in this way.

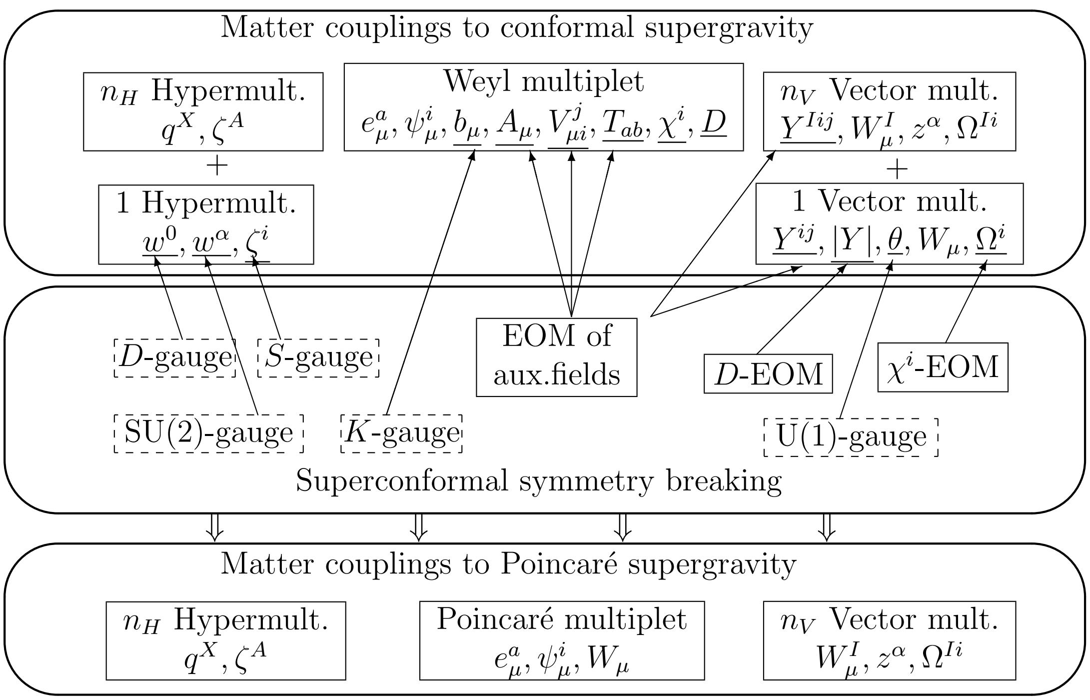
Similarly to the metric, also all other geometrical quantities of the big space can be decomposed in several parts, one of which defines the geometric quantities of the quaternionic-Kähler space. An important example is that in the connection $\hat{\omega}_{X\hat{A}}{}^{\hat{B}}$, there is a part $\omega_{Xi}{}^j$, which will play the role of SU(2) connection in the quaternionic-Kähler manifold. Such a connection was absent in the hyper-Kähler manifold, and is characteristic of quaternionic-Kähler manifolds.

5.5 The gauge fixing of the total action

We concentrate here on 4 dimensions. The principles are the same for $d = 5$ and for a large part also for $d = 6$.

The total action is the sum of the local superconformal action of the vector multiplet and the hypermultiplet. The former is (4.79), of which the most relevant terms have been rewritten in (4.81). The hypermultiplet action is (4.108). We assume that in both cases the geometric ingredients, the special Kähler geometry for the vector multiplets, and the quaternionic-Kähler geometry for the hypermultiplets, has been chosen such that there is in each sector one multiplet with negative kinetic energy and all the others have positive kinetic energies. The vector and hypermultiplet with negative kinetic energy will now be used for gauge fixing and for solving some equations of motion. Both actions contain also the fields of the Weyl multiplet. Dependent fields in this Weyl multiplet have already been solved for in the final actions that we presented. But there are also the auxiliary fields. This is the starting point for the figure 1, which can guide us through the following steps.

Figure 1: *Summary of the gauge-fixing procedure. The underlined terms of the upper box will be eliminated. The method that is used is indicated in the middle box. There, EOM stands for elimination by an equation of motion. Other fields are eliminated by gauge conditions. This leads to the field content indicated in the lower box.*



As mentioned in (5.2), the dilatation gauge field b_μ can be eliminated by a gauge choice for the special conformal transformations K. As well the Weyl multiplet as the vector multiplets contain moreover auxiliary fields (T_{ab}, $V_{\mu i}{}^j$ and Y^{ij}) which should be eliminated from the action by their respective field equations. The field equation of the auxiliary field

D is combined with the dilatation gauge choice to eliminate the modulus of the scalars of the compensating multiplets, $|Y|$ and w^0. One may choose the order in which these two equations are used. In the figure, we indicated to eliminate w^0 by the dilatation gauge choice and $|Y|$ by the D field equation for simplicity, but that can be interchanged at random. A similar remark holds for the elimination of the fermionic components of the compensating multiplets by the S-gauge and the use of the equation of motion of χ^i. The U(1) gauge eliminates the phase of the scalar of the compensating vector multiplet. Here is the only difference with 5 dimensions. In that case the superconformal algebra does not contain a U(1), but the scalar of the vector multiplet is also real, such that this step is not necessary. Similarly the quaternionic phases of the scalars of the compensating hypermultiplet are fixed by gauge choices of the SU(2). The only remaining field of the compensating multiplets is the vector field W_μ. This field becomes the graviphoton in the final super-Poincaré action. The result is the coupling of n_H hypermultiplets and n_V vector multiplets to $N = 2$, $d = 4$ Poincaré supergravity.

Remark that the fixing of gauge symmetries leads to a change in the definitions of covariant derivatives. Indeed, the fields that we use in the Poincaré supergravity are chosen such that they do not transform any more under the broken symmetries. Moreover, the parameters of the broken symmetries will be expressed in terms of the independent parameters by the so-called decomposition laws. The new transformation laws of the fields are obtained by replacing the parameters of the broken symmetries by means of these expressions.

We now give some details.

5.5.1 Equations of motion

The fields D and χ appear in the action as Lagrange multipliers. Their field equations imply the following conditions:

$$\begin{aligned} D: &\quad N + \tfrac{1}{4}k_D^2 = 0, \\ \chi^i: &\quad \tfrac{1}{2}\left(-N - \tfrac{1}{4}k^2\right)\gamma_a\psi_i^a + N_{IJ}\bar{X}^I\Omega_i^J + 2\mathrm{i}k_{D\hat{X}}\hat{f}^{\hat{X}}_{i\hat{A}}\zeta^{\hat{A}} = 0. \end{aligned} \tag{5.62}$$

The total action leads to the following equation of motion for the SU(2) gauge field $V^a{}_i{}^j$, and the other auxiliary fields Y^{ij} and T_{ab}:

$$\begin{aligned} V^a{}_i{}^j &= \frac{1}{2k_D^2}g_{\hat{X}\hat{Y}}\{\partial^a q^{\hat{X}} + gW^{aI}k_I^{\hat{X}}\}k^{\hat{Y}}{}_i{}^j \\ &\quad + \frac{1}{2k^2}\big[\tfrac{1}{12}(-N + 5k_D^2)\bar{\psi}_{bi}\gamma^{abc}\psi_c^j + \tfrac{1}{4}N_{IJ}\bar{\Omega}_i^I\gamma^a\Omega^{Jj} \\ &\quad + \tfrac{1}{3}N_{IJ}\bar{X}^I\bar{\Omega}_i^J\gamma^{ab}\psi_b^j - \tfrac{1}{3}\mathrm{i}k_{D\hat{X}}\hat{f}^{\hat{X}}_{i\hat{A}}\zeta^{\hat{A}}\gamma^{ab}\psi_b^j - \text{h.c.; traceless}\big], \qquad (5.63) \\ N_{IJ}Y^J_{ij} &= -\tfrac{1}{2}\mathrm{i}F_{IJK}\bar{\Omega}_i^J\Omega_j^K - 4gP_{Iij}. \\ \tfrac{1}{8}N_{IJ}X^IX^JT^+_{ab} &= \tfrac{1}{12}\left(-N - \tfrac{1}{4}k^2\right)\bar{\psi}_a^i\psi_b^j\varepsilon_{ij} + \tfrac{1}{6}\left(\tfrac{1}{2}\mathrm{i}k_{\hat{X}}f^{i\hat{A}\hat{X}}\bar{\zeta}_{\hat{A}} - N_{IJ}X^J\bar{\Omega}^{iI}\right)\gamma_a\psi_b^j\varepsilon_{ij} \\ &\quad + \tfrac{1}{4}N_{IJ}X^I\widehat{F}^{+J}_{ab} + \tfrac{1}{32}\mathrm{i}\bar{F}_{IJK}\bar{\Omega}^{iI}\gamma_{ab}\Omega^{jJ}X^K\varepsilon_{ij} + \tfrac{1}{8}\bar{\zeta}_{\hat{A}}\gamma_{ab}\zeta_{\hat{B}}d^{\hat{A}\hat{B}}. \end{aligned}$$

5.5.2 Gauge choices and decomposition laws

We have already commented on the gauge choice for the special conformal transformations, see (5.2). For the dilatations, we have already mentioned in the simple example (2.11) that our aim is to have a standard kinetic term for gravity. Hence, we just collect the terms with the scalar curvature and obtain (with units such that $\kappa^2 = 1$)

$$D\text{-gauge:} \qquad \tfrac{1}{6}N - \tfrac{1}{12}k_D^2 = \tfrac{1}{2}. \tag{5.64}$$

Combining this with the field equation for D, (5.62), we obtain

$$N = 1, \qquad k_D^2 = -4. \tag{5.65}$$

Using the basis (5.56) and (5.59) we thus have at the quaternionic side

$$w^0 = 1. \tag{5.66}$$

The condition that fixes the dilation gauge is physically the requirement that the kinetic terms of the scalars and the spin-2 field are not mixed. The total action does contain terms of the form $\gamma^{\mu\nu}\partial_\mu\psi_\nu^i$, multiplied with fermion fields of the vector and hypermultiplet. These imply thus a mixing of the kinetic terms of spin 3/2 and 1/2. We choose in analogy to the bosonic sector the S-gauge condition such that such a mixing does not occur. Hence, we put to zero the coefficient of $\gamma^{\mu\nu}\partial_\mu\psi_\nu^i$.

$$S\text{-gauge:} \qquad N_{IJ}\bar{X}^I\Omega_i^J - \mathrm{i}k_{D\hat{X}}\hat{f}^{\hat{X}}_{i\hat{A}}\zeta^{\hat{A}} = 0. \tag{5.67}$$

This coincides with the requirement that the D-gauge is invariant under ordinary supersymmetry, which simplifies the decomposition laws. Combining this with the field equation of the auxiliary field χ_i (5.62) leads to:

$$N_{IJ}\bar{X}^I\Omega_i^J = 0, \qquad k_{D\hat{X}}\hat{f}^{\hat{X}}_{i\hat{A}}\zeta^{\hat{A}} = 0. \tag{5.68}$$

In the coordinates (5.55) and (5.57) for the hypermultiplet, we have (see [53]):

$$k_{\hat{X}}\hat{f}^{\hat{X}}_{i\hat{A}} = 2w^0 g_{00}\hat{f}^0_{i\hat{A}} = 2\mathrm{i}\varepsilon_{ij}\delta^j_{\hat{A}}\sqrt{\tfrac{1}{2}w^0}. \tag{5.69}$$

Therefore, (5.68) says that

$$\zeta^i = 0. \tag{5.70}$$

We already discussed the geometry of the vector multiplet and used the U(1) gauge (5.30). The SU(2) allows us similarly to fix the phases of the compensating quaternion in the hypermultiplet. We choose w^α constant:

$$\mathrm{SU(2)\text{-}gauge:} \qquad w^\alpha = w_0^\alpha\,. \tag{5.71}$$

Decomposition rules. The above gauge choices imply that the parameters of the gauge-fixed symmetries are functions of the remaining gauge symmetries. We mentioned this already for the K parameter in (5.3). By the fact that the S gauge has been chosen to be the supersymmetry transformation of the D gauge implies that the parameter of the latter can just be put equal to zero.

$$\Lambda_D = 0. \tag{5.72}$$

There is a more complicated result for the parameter of the special supersymmetry, η. Varying the S gauge leads to

$$\eta_i = \tfrac{1}{4}\mathrm{i}F_{IJK}\bar{X}^K\Omega_j^I\bar{\epsilon}^j\Omega_i^J - \tfrac{1}{16}\bar{\zeta}^{\hat{A}}\gamma_{ab}\zeta^{\hat{B}}\gamma^{ab}\epsilon^j\varepsilon_{ij}d_{\hat{A}\hat{B}} - g\bar{X}^I P_{Iij}\epsilon^j - \tfrac{1}{4}N_{IJ}\bar{\Omega}^{jI}\gamma^a\Omega_i^J\gamma_a\epsilon_j. \tag{5.73}$$

Varying the condition (5.71) leads to an expression for the SU(2) parameter $U_i{}^j$:

$$U_i{}^j = 2\omega_{Xi}{}^j\left(\delta_G q^X + \delta_Q q^X\right) + 2g\alpha^I P_{Ii}{}^j. \tag{5.74}$$

This expression determines that any gauge symmetry has now a contribution from the SU(2) R-symmetry. This is an important fact in quaternionic geometry.

The gauge fixing of U(1) and the corresponding decomposition law was already given in (5.36).

5.5.3 Action for Poincaré supergravity.

We give here some important terms of the final result:

$$\begin{aligned} S \;=\; & \tfrac{1}{2}eR - \left(\partial_{\bar\beta}\partial_\alpha\mathcal{K}\right)\partial_\mu z^\alpha\,\partial^\mu z^{\bar\beta} - \tfrac{1}{16}\left(F_{IJ}+\bar{F}_{IJ}\right)e^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I F_{\rho\sigma}{}^J \\ & + \big(-\zeta_A \not{D}\zeta^B d^A{}_B - \tfrac{1}{4}N_{IJ}\bar{\Omega}^{iI}\not{D}\Omega_i^J - \tfrac{1}{2}\bar{\psi}_{ia}\gamma^{abc}\mathcal{D}_b\psi_c^i \\ & + \Big(-\tfrac{1}{8}N_{IJ}\widehat{F}_{\mu\nu}^{+I}\widehat{F}^{+\mu\nu J} + \text{h.c.}\Big) + \cdots \end{aligned} \tag{5.75}$$

Remark that the covariant derivatives are the previously introduced superconformal covariant derivatives, where the auxiliary fields have to be replaced by the values that they got from their field equations.

6 Special Kähler geometry.

6.1 Dualities in general.

A prerequisite to understand the following development, is a study of the symplectic transformations. These are the duality symmetries of 4 dimensions, the generalizations of the Maxwell dualities. They were first discussed in [73, 74, 75, 76]. Consider the kinetic terms of the vector fields, $\mathcal{L}_1$ in (5.32), for an arbitrary number (m) of vectors. $\mathcal{N}_{IJ}$ are coupling constants or functions of scalars. Remember that the complex conjugate of $\mathcal{F}^+$ is $\mathcal{F}^-$. Defining

$$G_{+I}^{\mu\nu} \equiv 2\mathrm{i}\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{\mu\nu}^{+I}} = \mathcal{N}_{IJ}\mathcal{F}^{+J\,\mu\nu}\,, \tag{6.1}$$

the Bianchi identities and field equations can be written as

$$\begin{aligned} \partial^\mu \operatorname{Im} \mathcal{F}^{+I}_{\mu\nu} &= 0 \qquad \text{Bianchi identities} \\ \partial_\mu \operatorname{Im} G^{\mu\nu}_{+I} &= 0 \qquad \text{Equations of motion.} \end{aligned} \tag{6.2}$$

This set of equations is invariant under $GL(2m, \mathbb{R})$:

$$\begin{pmatrix} \widetilde{\mathcal{F}}^+ \\ \widetilde{G}_+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} . \tag{6.3}$$

The $G_{\mu\nu}$ are related to the $F_{\mu\nu}$ as in (6.1). Now we limit the transformations to those that preserve such a relation. Therefore, we should have

$$\begin{aligned} &\widetilde{G}^+ = (C + D\mathcal{N})\mathcal{F}^+ = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\widetilde{\mathcal{F}}^+ \\ &\to \boxed{\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}} \qquad . \end{aligned} \tag{6.4}$$

As $\tilde{G}_{+\mu\nu}$ should be the derivative of a transformed action to describe the field equations, this requirement imposes that the matrix $\tilde{\mathcal{N}}$ should be symmetric. For a general $\mathcal{N}$ this implies (using rescalings of the field strengths)

$$A^T C - C^T A = 0 \quad , \quad B^T D - D^T B = 0 \quad , \quad A^T D - C^T B = \mathbb{1} . \tag{6.5}$$

These equations express that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R}) , \tag{6.6}$$

as the explicit condition is

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{6.7}$$

Thus the remaining transformations are real symplectic ones in dimension $2m$, where m is the number of vector fields.

A $2m$ component column V which transforms under the symplectic transformations as $\widetilde{V} = \mathcal{S}V$ is called a symplectic vector. The prime example is thus $V = \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix}$. The invariant inner product of two symplectic vectors V and W is

$$\langle V, W \rangle \equiv V^T \Omega W . \tag{6.8}$$

The symplectic transformations thus transform solutions of (6.2) into solutions. However, they are not invariances of the action. Indeed, writing

$$\mathcal{L}_1 = \tfrac{1}{2} \operatorname{Im} \left(\mathcal{N}_{IJ}\, F^{+I}_{\mu\nu} F^{+\mu\nu\, J} \right) = \tfrac{1}{2} \operatorname{Im} \left(F^{+I} G_{+I} \right) \ , \tag{6.9}$$

we obtain

$$\operatorname{Im}\widetilde{F}^{+I}\widetilde{G}_{+I} = \operatorname{Im}\left(F^+G_+\right) + \operatorname{Im}\left(2F^+(C^TB)G_+ + F^+(C^TA)F^+ + G_+(D^TB)G_+\right) \ . \quad (6.10)$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a four–divergence, as $\operatorname{Im} F^+F^+ = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and the matrices A and C are real. For $B \neq 0$ the Lagrangian is *not* invariant.

If sources are added to the equations (6.2), the transformations can be extended to the dyonic solutions of the field equations by letting the magnetic and electric charges $\begin{pmatrix} q^I_m \\ q_{e\,I}\end{pmatrix}$ transform as a symplectic vector too. The Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations to a discrete subgroup $\mathrm{Sp}(2m,\mathbb{Z})$.

Finally, the transformations with $B \neq 0$ will be non–perturbative. This can be seen from the fact that they do not leave the purely electric charges invariant, or from the fact that (6.4) shows that these transformations invert $\mathcal{N}$, which plays the role of the gauge coupling constant.

The important properties for the matrix $\mathcal{N}$ is that it should be symmetric and $\operatorname{Im}\mathcal{N} < 0$ in order to have positive kinetic terms. These properties are preserved under symplectic transformations defined by (6.4).

The simplest case is with one Abelian vector. Take $\mathcal{N} = S$, a complex field. The action is

$$\mathcal{L} = \tfrac{1}{4}(\operatorname{Im} S)F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{8}(\operatorname{Re} S)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \ . \quad (6.11)$$

The set of Bianchi identities and field equations is invariant under symplectic transformations, transforming the field S as

$$\widetilde{S} = \frac{C + DS}{A + BS} \qquad \text{where} \qquad AD - BC = 1 \, . \quad (6.12)$$

If the rest of the action, in particular the kinetic term for S, is also invariant under this transformation, then this is a symmetry. These transformations form an $\mathrm{Sp}(2;\mathbb{R}) = \mathrm{S}\ell(2,\mathbb{R})$ symmetry. The $\mathrm{S}\ell(2,\mathbb{Z})$ subgroup is generated by

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \widetilde{S} = S + 1 & \widetilde{S} = -\dfrac{1}{S} \ . \end{array} \quad (6.13)$$

Note that $\operatorname{Im} S$ is invariant in the first transformation, while the second one replaces $\operatorname{Im} S$ by its inverse. $\operatorname{Im} S$ is the coupling constant. Therefore, the second transformation, cannot be a perturbative symmetry. It relates the strong and weak coupling description of the theory.

For another example, namely S and T dualities in this framework, see [77].

6.2 Symplectic transformations in $N = 2$

In $N = 2$ the tensor $\mathcal{N}$ is determined by the function F as explained in section 5.3. The definition of $\mathcal{N}$ can be written in a clarifying way as follows

$$\partial_{\bar{\gamma}}\bar{F}_I = \mathcal{N}_{IJ}\partial_{\bar{\gamma}}\bar{X}^J\,, \qquad F_I = \mathcal{N}_{IJ}X^J\,. \tag{6.14}$$

One regards $(\partial_{\bar{\gamma}}\bar{F}_I, F_I)$ as an $n+1$ by $n+1$ matrix to see how this defines the matrix $\mathcal{N}$. From this definition it is easy to see that $\mathcal{N}$ transforms in the appropriate way if we define

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}\,, \qquad U_\alpha = \begin{pmatrix} \partial_\alpha X^I \\ \partial_\alpha F_I \end{pmatrix} \tag{6.15}$$

(and their complex conjugates) as symplectic vectors. They thus transform as in (6.3). With this identification in mind, we can reconsider the kinetic terms of the scalars. Then it is clear that the Kähler potential (5.33), and the constraint (5.22) are symplectic invariants. This will lead to a new formulation of special geometry in section 6.3.

The equation

$$\tilde{X}^I = A^I{}_J X^J + B^{IJ}F_J(X) \tag{6.16}$$

expresses the dependence of the new coordinates $\tilde{X}$ on the old coordinates X. If this transformation is invertible [24], the $\tilde{F}_I$ are again the derivatives of an new function $\tilde{F}(\tilde{X})$ of the new coordinates,

$$\tilde{F}_I(\tilde{X}) = \frac{\partial \tilde{F}(\tilde{X})}{\partial \tilde{X}^I}\,. \tag{6.17}$$

The integrability condition which implies this statement is equivalent to the condition that $\mathcal{S}$ is a symplectic matrix. In the supergravity case, one can obtain $\tilde{F}$ due to the homogeneity:

$$\tilde{F}(\tilde{X}(X)) = \frac{1}{2}V^T \begin{pmatrix} C^TA & C^TB \\ D^TA & D^TB \end{pmatrix} V\,. \tag{6.18}$$

Hence we obtain a new formulation of the theory, and thus of the target-space manifold, in terms of the function $\tilde{F}$.

We have to distinguish two situations:

1. The function $\tilde{F}(\tilde{X})$ is different from $F(\tilde{X})$. In that case the two functions describe equivalent classical field theories. We have a *pseudo symmetry.* These transformations are called symplectic reparametrizations [78]. Hence we may find a variety of descriptions of the same theory in terms of different functions F.

2. If a symplectic transformation leads to the same function F, then we are dealing with a *proper symmetry.* This invariance reflects itself in an isometry of the target-space manifold.

[24] The full symplectic matrix is always invertible, but this part may not be. In rigid supersymmetry, the invertibility of this transformation is necessary for the invertibility of $\mathcal{N}$ (due to the positive definiteness of the full metric), but in supergravity we may have that the $\tilde{X}^I$ do not form an independent set, and then $\tilde{F}$ cannot be defined. See below.

Henceforth these symmetries are called 'duality symmetries', as they are generically accompanied by duality transformations on the field equations and the Bianchi identities. The question remains whether the duality symmetries comprise all the isometries of the target space, i.e. whether

$$Iso(\text{scalar manifold}) \subset \mathrm{Sp}(2(n+1), \mathbb{R})\,. \tag{6.19}$$

We investigated this question in [79] for the very special Kähler manifolds, and found that in that case one does obtain the complete set of isometries from the symplectic transformations. For generic special Kähler manifolds no isometries have been found that are not induced by symplectic transformations, but on the other hand there is no proof that these do not exist.

E.g. the symplectic transformations with

$$\mathcal{S} = \begin{pmatrix} \mathbb{1} & 0 \\ C & \mathbb{1} \end{pmatrix} \tag{6.20}$$

do not change the X^A and give $\tilde{F} = F + \frac{1}{2} C_{AB} X^A X^B$. So these give proper symmetries for any symmetric matrix C_{AB}. The symmetry of C is required for $\mathcal{S}$ to be symplectic. From the explicit construction it is clear that such addition of real quadratic terms to the prepotential do not change the action. Therefore, we should also modify the distinction between pseudo symmetries and proper symmetries that we mentioned earlier. When the prepotential only changes in this way, then the corresponding symmetry is still a proper symmetry.

That the full supersymmetric theory allows such symplectic transformations can be seen also in another way. I mentioned before that the vector multiplets are chiral multiplets, with $w = 1$ that satisfy constraints (4.20). One of these constraints is the Bianchi identity for the vector. The functions $F_I(X)$ transform also in a chiral way under supersymmetry, and thus define also chiral multiplets, which have Weyl weight $w = 1$. Now it turns out that the same conditions on these multiplets are in fact the field equations. One of these is the field equations of the vectors. Thus the symplectic vectors V (6.24) are the lowest components of a symplectic vector of chiral multiplets. If these symplectic vectors satisfy the constraints, then this implies as well that we have vector multiplets, as the field equations. This is thus a supersymmetric generalization of the symplectic set-up of section 6.1. For the supergravity case this has been worked out in [80]. In this way, even the equations for models in a parametrization without prepotential can be obtained. Such situations will be explained shortly.

I will finish this section with an example of a manifold with isometries. Consider (5.46). If we apply the symplectic transformation

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \end{pmatrix} \tag{6.21}$$

one arrives, using (6.18), at (5.47). So this is a symplectic reparametrization, and shows the equivalence of the two forms of F as announced earlier.

On the other hand consider

$$\mathcal{S} = \begin{pmatrix} 1+3\epsilon & \mu & 0 & 0 \\ \lambda & 1+\epsilon & 0 & 2\mu/9 \\ 0 & 0 & 1-3\epsilon & -\lambda \\ 0 & -6\lambda & -\mu & 1-\epsilon \end{pmatrix} \tag{6.22}$$

for infinitesimal ϵ, μ, λ. Then F is invariant. On the scalar field $z = X^1/X^0$, the transformations act as

$$\delta z = \lambda - 2\epsilon z - \mu z^2/3 \; . \tag{6.23}$$

They form an $\mathrm{SU}(1,1)$ isometry group of the scalar manifold. The domain were the metric is positive definite is $\mathrm{Im}\, z > 0$. This shows the identification of the manifold as the coset space in (5.46), (5.47).

6.3 Characteristics of a special geometry.

Special Kähler geometry [1] is defined by the couplings of the scalars in the locally supersymmetric theory, i.e. in the coupled Einstein–Maxwell theory. For other applications one is interested in a definition of special geometry independent of supersymmetry. A first step in that direction has been taken by Strominger [66]. He had in mind the moduli spaces of Calabi–Yau spaces. His definition is already based on the symplectic structure, which we also have emphasized. However, being already in the context of Calabi–Yau moduli spaces, his definition of special Kähler geometry omitted some ingredients that are automatically present in any Calabi–Yau moduli space, but have to be included as necessary ingredients in a generic definition. Another important step was obtained in [81]. Before, special geometry was connected to the existence of a holomorphic prepotential function $F(z)$. The special Kähler manifolds were recognized as those for which the Kähler potential can be determined by this prepotential, in a way to be described below. However, in [81] it was found that one can have $N = 2$ supergravity models coupled to Maxwell multiplets such that there is no such prepotential. These models were constructed by applying a symplectic transformation to a model with prepotential. This fact raised new questions: are all the models without prepotential symplectic dual to models with a prepotential? Can one still define special Kähler geometry starting from the models with a prepotential? Is there a more convenient definition which does not involve this prepotential? These questions have been answered in [82], and are reviewed here.

6.3.1 Symplectic formulation

This is thus a reformulation of the geometry that we have seen in section 5.3, using the symplectic formalism. The dilatational gauge fixing (the fixing of r above), is done by the condition (5.22). This condition is chosen in order to decouple kinetic terms of the graviton from those of the scalars. Using again symplectic vectors

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \tag{6.24}$$

this can be written as the condition on the symplectic inner product:

$$< V, \bar{V} >= X^I \bar{F}_I - F_I \bar{X}^I = \mathrm{i}\,. \qquad (6.25)$$

To solve this condition, we define

$$V = \mathrm{e}^{\mathcal{K}(z,\bar{z})/2} v(z)\,, \qquad (6.26)$$

where $v(z)$ is a holomorphic symplectic vector,

$$v(z) = \begin{pmatrix} Z^I(z) \\ \frac{\partial}{\partial Z^I} F(Z) \end{pmatrix}. \qquad (6.27)$$

The upper components here are arbitrary functions (up to conditions for non-degeneracy), reflecting the freedom of choice of coordinates z^α. The Kähler potential is

$$\mathrm{e}^{-\mathcal{K}(z,\bar{z})} = -\mathrm{i}\langle v, \bar{v}\rangle. \qquad (6.28)$$

The kinetic matrix for the vectors is given by

$$\mathcal{N}_{IJ} = \begin{pmatrix} \mathcal{D}_{\bar{\alpha}} \bar{F}_I(\bar{X}) & F_I \end{pmatrix} \begin{pmatrix} \mathcal{D}_{\bar{\alpha}} \bar{X}^J & X^J \end{pmatrix}^{-1}\,, \qquad (6.29)$$

where the matrices are $(n+1) \times (n+1)$ and

$$\mathcal{D}_{\bar{\alpha}} \bar{F}_I(\bar{X}) = \partial_{\bar{\alpha}} \bar{F}_I(\bar{X}) + \tfrac{1}{2}(\partial_{\bar{\alpha}}\mathcal{K}) \bar{F}_I(\bar{X})\,, \qquad \mathcal{D}_{\bar{\alpha}} \bar{X}^J = \partial_{\bar{\alpha}} \bar{X}^J + \tfrac{1}{2}(\partial_{\bar{\alpha}}\mathcal{K}) \bar{X}^J\,. \qquad (6.30)$$

Before continuing with general statements, it is time for an example. Consider the prepotential

$$F = -\mathrm{i} X^0 X^1. \qquad (6.31)$$

This is a model with $n = 1$. There is thus just one coordinate z. One has to choose a parametrization to be used in the upper part of (6.27). Let us take a simple choice: $Z^0 = 1$ and $Z^1 = z$. The full symplectic vector is then (as e.g. $F_0(Z) = -\mathrm{i}Z^1(z)$)

$$v = \begin{pmatrix} Z^0 \\ Z^1 \\ F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -\mathrm{i}z \\ -\mathrm{i} \end{pmatrix}. \qquad (6.32)$$

The Kähler potential is then directly obtained from (6.28), determining the metric:

$$\mathrm{e}^{-\mathcal{K}} = 2(z + \bar{z})\,; \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} \mathcal{K} = (z + \bar{z})^{-2}\,. \qquad (6.33)$$

The kinetic matrix for the vectors is diagonal. From (6.29) follows

$$\mathcal{N} = \begin{pmatrix} -\mathrm{i}z & 0 \\ 0 & -\mathrm{i}\frac{1}{z} \end{pmatrix}. \qquad (6.34)$$

Therefore the action contains

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{2}\,\mathrm{Re}\left[z\left(F_{\mu\nu}^{+0}\right)^2 + z^{-1}\left(F_{\mu\nu}^{+1}\right)^2\right] . \tag{6.35}$$

The domain of positivity for both metrics is $\mathrm{Re}\,z > 0$.

Before continuing, let us extend the Kähler-covariant derivatives introduced in (6.30) for future use:

$$\begin{aligned}
\mathcal{D}_\alpha V &= \partial_\alpha V + \tfrac{1}{2}(\partial_\alpha\mathcal{K})V, &\qquad \mathcal{D}_{\bar\alpha} V = \partial_{\bar\alpha} V - \tfrac{1}{2}(\partial_{\bar\alpha}\mathcal{K})V = 0,\\
\mathcal{D}_{\bar\alpha} \bar V &= \partial_{\bar\alpha} \bar V + \tfrac{1}{2}(\partial_{\bar\alpha}\mathcal{K})\bar V, &\qquad \mathcal{D}_{\alpha} \bar V = \partial_{\alpha} \bar V - \tfrac{1}{2}(\partial_{\alpha}\mathcal{K})\bar V = 0,\\
\mathcal{D}_\alpha v &= \partial_\alpha v + \ (\partial_\alpha\mathcal{K})v, &\qquad \mathcal{D}_{\bar\alpha} v = \partial_{\bar\alpha} v = 0,\\
\mathcal{D}_{\bar\alpha} \bar v &= \partial_{\bar\alpha} \bar v + \ (\partial_{\bar\alpha}\mathcal{K})\bar v, &\qquad \mathcal{D}_{\alpha} \bar v = \partial_{\alpha} \bar v = 0,
\end{aligned} \tag{6.36}$$

where the indicated vanishing of expressions is due to the definitions. We also use definitions

$$U_\alpha \equiv \mathcal{D}_\alpha V = \mathrm{e}^{\mathcal{K}/2}\mathcal{D}_\alpha v, \qquad \bar U_{\bar\alpha} \equiv \mathcal{D}_{\bar\alpha}\bar V = \mathrm{e}^{\mathcal{K}/2}\mathcal{D}_{\bar\alpha}\bar v. \tag{6.37}$$

They satisfy e.g.

$$\langle U_\alpha, \bar U_{\bar\beta}\rangle = -\mathrm{i} g_{\alpha\bar\beta}. \tag{6.38}$$

Exercise 6.1: These covariant derivatives are covariant for the remaining Kähler transformations after the decomposition law (5.36). Check this statement.

6.3.2 Absence of a prepotential in examples

The tensor calculus lead to the formulation of special geometry in terms of a prepotential $F(X)$. In the above symplectic formulation appears then its derivatives $F_I(X) = \frac{\partial}{\partial X^I}F(X)$. The latter F_I turn out to be more fundamental than $F(X)$ itself. We will show now that one can take alternative symplectic bases where F_I are not derivatives of a scalar prepotential $F(X)$. We will show this first by an example, and then give some general properties before exhibiting another class of examples that are relevant in string theory.

Let us come back to the example (6.31), on which we perform a symplectic mapping:

$$\widetilde{v} = \mathcal{S}v = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} v = \begin{pmatrix} 1\\ \mathrm{i}\\ -\mathrm{i}z\\ z\end{pmatrix} . \tag{6.39}$$

After this mapping, z is not any more a good coordinate for $(\widetilde{Z}^0, \widetilde{Z}^1)$, the upper two components of the symplectic vector z. This means that the symplectic vector cannot be obtained from a prepotential. We cannot obtain the symplectic vector from a form (6.27). No function $\widetilde{F}(\widetilde{Z}^0, \widetilde{Z}^1)$ exists.

The Kähler metric is still the same, (6.33), and one can again compute the vector kinetic matrix, either directly from (6.29), as the denominator is still invertible, or from (6.4):

$$\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} = -\mathrm{i}X^1(X^0)^{-1}\mathbb{1} = -\mathrm{i}z\mathbb{1}\,. \tag{6.40}$$

In this parametrization, the action is thus

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{2}\,\mathrm{Re}\left[z\left(F_{\mu\nu}^{+0}\right)^2 + z\left(F_{\mu\nu}^{+1}\right)^2\right]\,. \tag{6.41}$$

This action is not the same as the one before, but is a 'dual formulation' of the same theory, being obtained from (6.35) by a duality transformation. The straightforward construction in superspace or superconformal tensor calculus does not allow to construct actions without a superpotential. However, in [80] it has been shown that the field equations of these models can also be obtained from the superconformal tensor calculus. One just has to give up the concept of a superconformal invariant action.

Going to a dual formulation, one obtains a formulation with different symmetries in perturbation theory. The example that we used here appears in a reduction to $N = 2$ of two versions of $N = 4$ supergravity, known respectively as the 'SO(4) formulation' [83, 84, 85] and the 'SU(4) formulation' of pure $N = 4$ supergravity [86].

Finally let us note that we still could apply (6.29) because the matrix

$$\begin{pmatrix} X^I & \mathcal{D}_\alpha \bar{X}^I \end{pmatrix} \tag{6.42}$$

is always invertible if the metric $g_{\alpha\bar{\alpha}} = \partial_\alpha \partial_{\bar{\alpha}} K(z, \bar{z})$ is positive definite. Therefore, the inverse exists, and $\mathcal{N}_{IJ}$ can be constructed. However, the matrix

$$\begin{pmatrix} X^I & \mathcal{D}_\alpha X^I \end{pmatrix} \tag{6.43}$$

is not invertible in the formulation (6.39). **If** that matrix is invertible, then a prepotential exists [82].

It turns out that the following 3 conditions are equivalent:

1. (6.43) invertible;
2. special coordinates are possible;
3. a prepotential $F(X)$ exists.

An important example of the phenomenon of the absence of a prepotential is the description of the manifold

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(r,2)}{\mathrm{SO}(r)\otimes \mathrm{SO}(2)}\,. \tag{6.44}$$

This is the only special Kähler manifold that is a product of two factors [87]. Therefore it is also determined that in the classical limit of the compactified heterotic string, where the dilaton does not mix with the scalars of the other vector multiplets, the target space should have that form. The first formulation of these spaces used a function F of the form [67]

$$F = \frac{d_{ABC}X^A X^B X^C}{X^0} \ . \tag{6.45}$$

In fact, such a form of F is what one expects for all couplings which can be obtained from $d = 5$ supergravity [88]. Such manifolds are called 'very special Kähler manifolds'. In such a formulation for (6.44) the $\mathrm{SO}(r, 2)$ part of the duality group sits not completely in the perturbative part of the duality group, i.e. one needs $B \neq 0$ in the duality group to get the full $\mathrm{SO}(2, r)$. However, from the superstring compactification one expects $\mathrm{SO}(2, r; \mathbb{Z})$ as a perturbative (T-duality) group.

By making a symplectic transformation this can indeed be obtained [81]. After that symplectic transformation one has a symplectic vector (X^I, F_I) satisfying

$$X^I \,\eta_{IJ}\, X^J = 0 \ ; \qquad F_I = S\,\eta_{IJ}\, X^J \ , \tag{6.46}$$

where η_{IJ} is a metric for $\mathrm{SO}(2, r)$. The first constraint comes on top of the constraint (5.22), and thus implies that the variables z cannot be chosen between the X^I only. Indeed, S occurs only in F_I.

6.3.3 Definitions

After this extension of the formulation, the reader may ask the question what is then really special Kähler geometry. This was the question put in [82], and lead to some definitions. We will give here first a definitions using the prepotential, and then a second one using only the symplectic vectors. We will discuss then the equivalence. We will afterwards discuss a more mathematically inspired definition [89].

Definition 1 of (local) special Kähler geometry.

A special Kähler manifold is an n-dimensional Hodge–Kähler manifold with on any chart $n+1$ holomorphic functions $Z^I(z)$ and a holomorphic function $F(Z)$, homogeneous of second degree, such that, with (6.27), the Kähler potential is given by

$$e^{-\mathcal{K}(z,\bar{z})} = -\mathrm{i}\langle v, \bar{v}\rangle \,, \tag{6.47}$$

and on overlap of charts, the $v(z)$ are connected by symplectic transformations $\mathrm{Sp}(2(n+1), \mathbb{R})$ and/or Kähler transformations.

$$v(z) \to e^{f(z)}\mathcal{S}v(z) \,. \tag{6.48}$$

The transition functions should satisfy the cocycle condition on overlap of regions of three charts.

In the examples of section 6.3.2 this definition turns out not to be applicable in an arbitrary symplectic frame. Therefore we will not give a second definition, but we will then comment how they are anyway equivalent.

Definition 2 of (local) special Kähler geometry.

A special Kähler manifold is an n-dimensional Hodge–Kähler manifold, that is the base manifold of a $\mathrm{Sp}(2(n+1))\times \mathrm{U}(1)$ bundle. There should exist a holomorphic section $v(z)$ such that the Kähler potential can be written as (6.47) and it should satisfy the condition

$$\langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v\rangle = 0\,. \tag{6.49}$$

Note that the latter condition guarantees the symmetry of $\mathcal{N}_{IJ}$. This condition did not appear in [66], where the author had in mind Calabi–Yau manifolds. As we will see below, in those applications, this condition is automatically fulfilled. For $n>1$ the condition can be replaced by the equivalent condition

$$\langle \mathcal{D}_\alpha v, v\rangle = 0\,. \tag{6.50}$$

For $n=1$, the condition (6.49) is empty, while (6.50) is not. In [80] it has been shown that models with $n=1$ not satisfying (6.50) can be formulated.

Equivalence of the two definitions.

It is thus legitimate to ask about the equivalence of the two definitions. Indeed, we saw that in some cases definition 2 is satisfied, but one cannot obtain a prepotential F. However, that example, as others in [81], was obtained from performing a symplectic transformation from a formulation where the prepotential does exist. In [82] it was shown that this is true in general. If definition 2 is applicable, then there exists a symplectic transformation to a basis such that $F(Z)$ exists. However, in the way physical problems are handled, the existence of formulations without prepotentials can play an important role. For example, these formulations were used to prove that one can break $N=2$ supersymmetry partially to $N=1$ [90] and not necessarily to $N=0$, as it was thought before. This is an extremely important property for phenomenological applications.

On page 91, we gave equivalent conditions for the symplectic basis such that there exists a prepotential. The last of these statements can now be shown how this follows from the second definition. One first notices that (X^I/X^0) is independent of $\bar z$, such that from the functions $F_I(z,\bar z)$, one can define holomorphic functions

$$F_I(X) \equiv X^0 \frac{F_I}{X^0}\left(z\left(\frac{X^A}{X^0}\right)\right)\,. \tag{6.51}$$

The constraints (6.49) then imply

$$\begin{pmatrix} X^I \\ \partial_\alpha X^I\end{pmatrix} \partial_{[I}F_{J]}\ \begin{pmatrix} X^J & \partial_\alpha X^J\end{pmatrix} = 0\,, \tag{6.52}$$

from which it follows that in any patch $F_J = \frac{\partial}{\partial X^J}F(X)$ for some $F(X)$.

6.3.4 Symplectic equations and the curvature tensor

Let us first summarize the symplectic inner products that we found. They can be simply written in terms of the $2(n+1)\times 2(n+1)$ matrix

$$\mathcal{V} = \begin{pmatrix} U_\alpha & \bar V & \bar U^\alpha & V\end{pmatrix}, \tag{6.53}$$

where $\bar{U}^\alpha = g^{\alpha\bar{\beta}}\bar{U}_{\bar{\beta}}$, as

$$\mathcal{V}^T\Omega\mathcal{V} = \mathrm{i}\Omega. \tag{6.54}$$

Thus, the matrix $\mathcal{V}$ is also invertible. Covariant derivatives on these equations lead to new ones, like

$$\begin{aligned}&\langle V, \mathcal{D}_\alpha U_\beta\rangle = \langle \bar{V}, \mathcal{D}_\alpha U_\beta\rangle = \langle \bar{U}_{\bar{\gamma}}, \mathcal{D}_\alpha U_\beta\rangle = 0,\\ &\langle \mathcal{D}_\alpha U_\beta, U_\gamma\rangle + \langle U_\beta, \mathcal{D}_\alpha U_\gamma\rangle = 0.\end{aligned} \tag{6.55}$$

Note that $\mathcal{D}_\alpha U_\beta$ contains also Levi-Civita connection. Due to these equations and the invertibility of (6.53), $\mathcal{D}_\alpha U_\beta$ must be proportional to a $\bar{U}^\gamma$, and, as from its definition it is already symmetric in $(\alpha\beta)$, we can define a third rank symmetric tensor

$$\mathcal{D}_\alpha U_\beta = C_{\alpha\beta\gamma}\bar{U}^\gamma, \qquad C_{\alpha\beta\gamma} \equiv -\mathrm{i}\langle \mathcal{D}_\alpha U_\beta, U_\gamma\rangle. \tag{6.56}$$

Exercise 6.2: Check that if there is a prepotential, then one can write

$$C_{\alpha\beta\gamma} = \mathrm{i}F_{IJK}\mathcal{D}_\alpha X^I \mathcal{D}_\beta X^J \mathcal{D}_\gamma X^K, \tag{6.57}$$

where the covariant derivatives may be replaced by ordinary derivatives due to (4.70).

The curvature of the manifold can then be obtained from the commutator of covariant derivatives. As mentioned already, the connections contain Levi-Civita and Kähler connections, so the commutator will lead to a sum of the curvature of the Kähler manifold as function of the Levi-Civita connection of $g_{\alpha\bar{\beta}}$ and the curvature of the Kähler connection. The latter is the original U(1) from the superconformal algebra when it is pulled back to the spacetime. It acts as $\delta V = -\mathrm{i}\Lambda_A V$, and as one can see in (6.36), its gauge field has as (holomorphic, antiholomorphic) components $(-\frac{1}{2}\mathrm{i}\partial_\alpha\mathcal{K}, \frac{1}{2}\mathrm{i}\partial_{\bar{\alpha}}K)$. Its curvature is therefore $R^{\mathrm{K}}_{\alpha\bar{\beta}} = \mathrm{i}g_{\alpha\bar{\beta}}$. Therefore, the commutator of covariant derivatives should be of the form

$$\left[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\bar{\beta}}\right] V = \mathrm{i}R^{\mathrm{K}}_{\alpha\bar{\beta}}V = -g_{\alpha\bar{\beta}}V, \qquad \text{hence} \qquad -\bar{\mathcal{D}}_{\bar{\beta}}U_\alpha = -g_{\alpha\bar{\beta}}V. \tag{6.58}$$

On U_γ we get (see e.g. (2.60) and (A.5) for conventions)

$$\left[\mathcal{D}_\alpha, \mathcal{D}_{\bar{\beta}}\right] U_\gamma = \mathrm{i}R^{\mathrm{K}}_{\alpha\bar{\beta}}U_\gamma - R^\delta{}_{\gamma\alpha\bar{\beta}}U_\delta. \tag{6.59}$$

To make the calculation of the left-hand side, one lemma that we still need is that $C_{\alpha\beta\gamma}$ is covariantly holomorphic:

$$\mathrm{i}\bar{\mathcal{D}}_{\bar{\delta}}C_{\alpha\beta\gamma} = \langle \bar{\mathcal{D}}_{\bar{\delta}}\mathcal{D}_\alpha U_\beta, U_\gamma\rangle + \langle \mathcal{D}_\alpha U_\beta, \bar{\mathcal{D}}_{\bar{\delta}}U_\gamma\rangle = 0. \tag{6.60}$$

Both terms vanish separately. For the first term, use (6.59) and (6.58) and e.g. $\langle U_\alpha, U_\beta\rangle = 0$. The second term is by these relations immediately zero.

Now everything is ready for a short proof. With (6.56) and this covariant holomorphicity of C we get

$$\bar{\mathcal{D}}_{\bar{\beta}}\mathcal{D}_\alpha U_\gamma = C_{\alpha\gamma\epsilon} g^{\epsilon\bar{\epsilon}} \bar{C}_{\bar{\epsilon}\bar{\beta}\bar{\delta}} g^{\bar{\delta}\delta} U_\delta. \tag{6.61}$$

On the other hand

$$\mathcal{D}_\alpha \bar{\mathcal{D}}_{\bar{\beta}} U_\gamma = g_{\gamma\bar{\beta}} U_\alpha. \tag{6.62}$$

Subtracting the two gives

$$- R^\delta{}_{\gamma\alpha\bar{\beta}} U_\delta - g_{\alpha\bar{\beta}} \delta^\delta_\gamma U_\delta = g_{\gamma\bar{\beta}} \delta^\delta_\alpha U_\delta - C_{\alpha\gamma\epsilon} g^{\epsilon\bar{\epsilon}} \bar{C}_{\bar{\epsilon}\bar{\beta}\bar{\delta}} g^{\bar{\delta}\delta} U_\delta. \tag{6.63}$$

We can drop the U_δ, e.g. by taking a symplectic product of this relation with $\bar{U}^\phi$. This establishes the form of the curvature tensor [67]

$$R^\delta{}_{\beta\alpha\bar{\gamma}} = -g_{\alpha\bar{\gamma}} \delta^\delta_\beta - g_{\beta\bar{\gamma}} \delta^\delta_\alpha + C_{\alpha\beta\epsilon} g^{\epsilon\bar{\epsilon}} \bar{C}_{\bar{\epsilon}\bar{\beta}\bar{\delta}} g^{\bar{\delta}\delta}. \tag{6.64}$$

Exercise 6.3: Check that this can also be written as

$$R^\alpha_{\beta\gamma}{}^\delta = -\delta_\beta{}^\alpha \delta_\gamma{}^\delta - \delta_\gamma{}^\alpha \delta_\beta{}^\delta + \mathrm{e}^{2\mathcal{K}} \mathcal{W}_{\beta\gamma\epsilon} \bar{\mathcal{W}}^{\epsilon\alpha\delta}\,, \tag{6.65}$$

where

$$\mathcal{W}_{\alpha\beta\gamma} = \mathrm{i} F_{IJK}\big(Z(z)\big) \frac{\partial Z^I}{\partial z^\alpha} \frac{\partial Z^J}{\partial z^\beta} \frac{\partial Z^K}{\partial z^\gamma}\,. \tag{6.66}$$

Having all this machinery, we want to derive a few more relations that are often used in special Kähler geometry.

Define the $2(n+1) \times (n+1)$ matrix $\mathcal{U}$ as the left part of $\mathcal{V}$, (6.53),

$$\mathcal{U} = \begin{pmatrix} U_\alpha & \bar{V} \end{pmatrix} = \begin{pmatrix} \mathcal{D}_\alpha X^I & \bar{X}^I \\ \mathcal{D}_\alpha F_I & \bar{F}_I \end{pmatrix} = \begin{pmatrix} x^I \\ f_I \end{pmatrix}, \tag{6.67}$$

where the last equation defines $(n+1) \times (n+1)$ matrices. The matrix x^I is invertible. Then the definition of $\mathcal{N}$ in (6.29) is now written as

$$\mathcal{N}_{IJ} = \bar{f}_I (\bar{x})^{-1}_J. \tag{6.68}$$

This leads to

$$\mathcal{U} = \begin{pmatrix} 1 \\ \mathcal{N} \end{pmatrix} x. \tag{6.69}$$

Check that the dimensions fit for this matrix equation to make sense. Now we introduce the $(n+1) \times (n+1)$ matrix

$$G = \begin{pmatrix} g_{\alpha\bar{\beta}} & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{i} \mathcal{U}^T \Omega \bar{\mathcal{U}}, \tag{6.70}$$

with Ω the symplectic metric as in (6.7). This equation holds due to the gauge fixing and definitions of the metric in special geometry. Combining the previous equations leads to $G=-2x^T(\mathrm{Im}\,\mathcal{N})\bar{x}$, or in full

$$\begin{pmatrix} g_{\alpha\bar\beta} & 0\\ 0 & 1\end{pmatrix} = \mathrm{i}\begin{pmatrix}\langle U_\alpha,\bar U_{\bar\beta}\rangle & \langle U_\alpha, V\rangle\\ \langle \bar V,\bar U_{\bar\beta}\rangle & \langle \bar V, V\rangle\end{pmatrix} = -2\begin{pmatrix}\mathcal{D}_\alpha X^I\\ \bar X^I\end{pmatrix}\mathrm{Im}\,\mathcal{N}_{IJ}\begin{pmatrix}\mathcal{D}_{\bar\beta}\bar X^J & X^J\end{pmatrix}. \tag{6.71}$$

The following consequence is often used:

$$-\tfrac{1}{2}(\mathrm{Im}\,\mathcal{N})^{-1|IJ} = \mathcal{D}_\alpha X^I g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar X^J + \bar X^I X^J. \tag{6.72}$$

Another immediate consequence of these equations is

$$\bar{\mathcal{U}}G^{-1}\mathcal{U}^T = \bar U_{\bar\beta} g^{\bar\beta\alpha}U_\alpha^T + V\bar V^T = -\frac{1}{2}\begin{pmatrix}1\\ \mathcal{N}\end{pmatrix}(\mathrm{Im}\,\mathcal{N})^{-1}\begin{pmatrix}1 & \bar{\mathcal{N}}\end{pmatrix} = -\frac{1}{2}\mathcal{M} + \frac{1}{2}\mathrm{i}\Omega, \tag{6.73}$$

in terms of the matrix $\mathcal{M}$ introduced in [91]:

$$\mathcal{M} \equiv \begin{pmatrix}\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\,\mathcal{N}^{-1}\,\mathrm{Re}\,\mathcal{N}\\ \mathrm{Re}\,\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\,\mathcal{N} + \mathrm{Re}\,\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1}\,\mathrm{Re}\,\mathcal{N}\end{pmatrix}. \tag{6.74}$$

6.3.5 A mathematically inspired definition

We now present the definition that has been given by Freed [89]. We start by defining 'rigid special Kähler geometry', that is the geometry of the scalars in rigid vector multiplets. In the mathematics literature this is called 'affine special Kähler geometry'. Then special Kähler geometry for supergravity is called 'projective special Kähler geometry', because it is defined from the former as a projectivization, which is in fact the gauge fixing of dilatations and the U(1) from the superconformal group.

Definition of rigid special Kähler geometry.

The definition can be shortly formulated as

> *A rigid special Kähler manifold is a Kähler manifold with Kähler form Φ and complex structure J such that there is a real, flat, torsion free, symplectic connection ∇ that preserves J seen as a 1-form, that is*
>
> $$\nabla\left((\mathrm{d}\phi^i)J_i{}^j\right) = 0, \qquad i,j = 1,\ldots,2n, \tag{6.75}$$
>
> *where ϕ^i denote the (real) coordinates of the manifold.*

Now we analyse this definition to see the relation with what has been presented before. First, it is mentioned that we start from a Kähler manifold. This means that the manifold as real manifold has even dimension. There is a hermitian metric with Levi-Civita connection D_i such that $D_iJ_j{}^k = 0$. We can take then complex coordinates z^α, $\bar z^{\bar\alpha}$, with $\alpha,\bar\alpha = 1,\ldots,n$, such that the metric has only non-zero coefficients $g_{\alpha\bar\beta} = g_{\bar\beta\alpha}$, and in these coordinates the complex structure has the form

$$J_\alpha{}^\beta = \mathrm{i}\delta_\alpha^\beta, \qquad J_{\bar\alpha}{}^{\bar\beta} = -\mathrm{i}\delta_{\bar\alpha}{}^{\bar\beta}, \qquad J_\alpha{}^{\bar\beta} = J_{\bar\alpha}{}^\beta = 0. \tag{6.76}$$

The Kähler 2-form is then defined as

$$\Phi = \tfrac{1}{2} J_i{}^k g_{kj} \mathrm{d}\phi^i \wedge \mathrm{d}\phi^j = \mathrm{i} g_{\alpha\bar\beta} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar z^{\bar\beta} \,. \tag{6.77}$$

This defines a symplectic structure on the Kähler manifold.

Then the definition involves another connection ∇. It is mentioned that it is flat, i.e. its curvature tensor vanishes. If the connection components of this connection are denoted as $A_{ij}{}^k$, the conditions are

$$\begin{aligned} \text{torsionless} \;&:\; A_{ij}{}^k = A_{ji}{}^k \,, \\ \text{symplectic} \;&:\; \partial_i J_{jk} + 2A_{i[j}{}^\ell J_{k]\ell} = 0 \,, \\ \text{flat} \;&:\; \partial_{[i} A_{j]k}{}^\ell + A_{m[i}{}^\ell A_{j]k}{}^m = 0 \,, \\ (6.75) \;&:\; \partial_{[i} J_{j]}{}^k + A_{\ell[i}{}^k J_{j]}{}^\ell = 0 \,. \end{aligned} \tag{6.78}$$

The statement that it is a symplectic connection means that $\nabla\Phi = 0$. These statements imply that there is a 'flat frame' such that the Kähler 2-form takes the form (the $-$ sign is for convenience, to agree with normalizations already chosen)

$$\Phi = -\mathrm{d}x^I \wedge \mathrm{d}f_I, \qquad I = 1, \ldots, n, \tag{6.79}$$

where we have denote the real coordinates in this frame as x^I and f_I. As the connection is also torsionless, it means that

$$\nabla \mathrm{d}\phi^i = A^i_{jk} \mathrm{d}\phi^j \wedge \mathrm{d}\phi^k = 0 \,, \tag{6.80}$$

when we denote the connection coefficients as A^i_{jk}. Combining this with (6.75) gives that the projection to holomorphic quantities is closed under ∇:

$$\nabla \pi^i_{(1,0)} = 0 \,, \qquad \pi^i_{(1,0)} = \tfrac{1}{2} \left[\mathrm{d}\phi^i + (\mathrm{d}\phi^j) J_j{}^i \right] . \tag{6.81}$$

As ∇ is flat, this means that $\pi^i_{(1,0)}$ is also ∇-exact. In the flat frame ∇ connection reduces to an ordinary differential. Therefore we can write (splitting an upper index i in an upper I and a lower I)

$$\pi^I_{(1,0)} = \mathrm{d}X^I \,, \qquad \pi_{(1,0)I} = \mathrm{d}F_I \,, \tag{6.82}$$

where in the complex frame, X^I and F_I are holomorphic, as one can easily verify taking an antiholomorphic derivative and using (6.76). Obviously, one can repeat the same for the projections to antiholomorphic quantities, and this leads to

$$\mathrm{d}x^I = \mathrm{d}X^I(z) + \mathrm{d}\bar X^I(\bar z) \,, \qquad \mathrm{d}f_I = \mathrm{d}F_I(z) + \mathrm{d}\bar F_I(\bar z) \,. \tag{6.83}$$

We now compare (6.77) and (6.79) using these expressions, leading to

$$-\Phi = -\mathrm{i} g_{\alpha\bar\beta} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar z^{\bar\beta} = \mathrm{d}X^I \wedge \mathrm{d}F_I + \mathrm{d}X^I \wedge \mathrm{d}\bar F_I + \mathrm{d}\bar X^I \wedge \mathrm{d}F_I + \mathrm{d}\bar X^I \wedge \mathrm{d}\bar F_I \,. \tag{6.84}$$

The first term in the last expression is quadratic in holomorphic variables, and should thus vanish by itself:

$$\mathrm{d}X^I(z) \wedge \mathrm{d}F_I(z) = 0 \,. \tag{6.85}$$

This is similar to the statement (6.49) that we saw before in a definition of special geometry. The last term is similar for the antiholomorphic parts. The metric is thus obtained as

$$g_{\alpha\bar\beta} = \mathrm{i}\partial_\alpha\partial_{\bar\beta}\left(X^I\bar F_I - F_I\bar X^I\right)\,. \tag{6.86}$$

It can be shown that if this metric is positive definite, $\partial_\alpha X^I$ should be invertible [82]. Therefore, in this case $X^I(z)$ can be inverted, and hence also $F_I(z)$ can be written as $F_I(X(z))$. Therefore we have then

$$\mathrm{d}F_I(z) = \frac{\partial F_I(X)}{\partial X^J}\mathrm{d}X^J(z)\,, \tag{6.87}$$

and (6.85) becomes the integrability condition that

$$F_I(X) = \frac{\partial F(X)}{\partial X^I}\,, \tag{6.88}$$

for some scalar $F(X)$. This is the prepotential function from which we started the derivation of the action of vector multiplets in section 4.3.1. We then obtain that in rigid special geometry if $\frac{\partial X^I}{\partial z^\alpha}$ is invertible, the metric is

$$g_{\alpha\bar\beta} = \mathrm{i}\left(\bar F_{IJ} - F_{IJ}\right)\frac{\partial X^I}{\partial z^\alpha}\frac{\partial \bar X^I}{\partial \bar z^{\bar\beta}}\,,\qquad F_{IJ} = \frac{\partial^2 F(X)}{\partial X^I\partial X^J}\,, \tag{6.89}$$

in agreement with (4.79).

When one turns from the ∇-flat basis to the basis z^α, $\bar z^{\bar\alpha}$, the non-vanishing connection coefficients of ∇ are

$$A_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma\,,\qquad A_{\alpha\beta}^{\bar\gamma} = g^{\bar\gamma\gamma}C_{\alpha\beta\gamma}(z)\,, \tag{6.90}$$

and their complex conjugates, where $\Gamma_{\alpha\beta}^\gamma$ are the Levi-Civita connections for the Kähler metric

$$\Gamma_{\alpha\beta}^\gamma = g^{\gamma\bar\gamma}\partial_\alpha g_{\beta\bar\gamma}\,, \tag{6.91}$$

and $C_{\alpha\beta\gamma}(z)$ are defined by (6.90). One can prove that they are holomorphic, completely symmetric and that

$$D_{[\delta}C_{\alpha]\beta\gamma} = \partial_{[\delta}C_{\alpha]\beta\gamma} - \Gamma^\epsilon_{\beta[\delta}C_{\alpha]\epsilon\gamma} - \Gamma^\epsilon_{\gamma[\delta}C_{\alpha]\beta\epsilon} = 0\,. \tag{6.92}$$

The curvature tensor is then similar to (6.64)

$$R^\delta{}_{\beta\alpha\bar\gamma}(\Gamma) = C_{\alpha\beta\epsilon}g^{\epsilon\bar\epsilon}\bar C_{\bar\epsilon\bar\beta\bar\delta}g^{\bar\delta\delta}\,, \tag{6.93}$$

in order that A in (6.90) leads to a vanishing curvature. In case that the symplectic basis allows a prepotential, we have

$$C_{\alpha\beta\gamma} = \mathrm{i}F_{IJK}\partial_\alpha X^I\partial_\beta X^J\partial_\gamma X^K\,. \tag{6.94}$$

Conversely, one can show that if a Kähler manifold has a curvature tensor of the form (6.93) where the coefficients $C_{\alpha\beta\gamma}$ are holomorphic, symmetric and satisfy (6.92), then one can construct the connection ∇ with the properties as in the definition that we give above. Hence, in that case the manifold is a rigid special Kähler manifold.

Definition of (projective) special Kähler geometry.

The extra ingredient with respect to rigid special Kähler geometry is the closed homothetic Killing vector, i.e. a vector satisfying (2.30). In presence of a complex structure that is preserved by the connection, this implies the presence of an isometry, which is $k_D^j J_j{}^i$. These two operations are related to the dilatations and U(1) transformations in the way that we developed special geometry from superconformal methods.

The first part, defining a projective Kähler manifold, is independent of whether we are discussing special geometry. Hence, this would apply also for $N = 1$.

We demand that the existence of a holomorphic closed homothetic Killing vector on a Kähler manifold $\tilde{\mathcal{M}}$. This means that there is a vector with components $H^I(X)$ and their complex conjugates, where X^I denote the holomorphic coordinates, with $D_I H^J = \delta_I^J$. The action of H generates equivalence classes. The projective Kähler manifold is the manifold of equivalence classes $\mathcal{M} = \tilde{\mathcal{M}}/\mathbb{C}^\times$ if $\mathbb{C}^\times$ denotes the action by the operation generated by H. This is a Kähler manifold too. To prove this, one first chooses homogeneous coordinates such that $H^I = X^I$. One can compare this now with start of section 5.1, where this Kähler potential was denoted as $-N$. The condition that this vector is a closed homothetic Killing vector amounts to the second equation in (5.6). where $N_{I\bar{J}}$ is the Kähler metric in $\tilde{\mathcal{M}}$. After a suitable Kähler transformation, this implies that the Kähler potential $\tilde{K} = -N$ from which $N_{I\bar{J}}$ is derived, is homogeneous of first degree both in X and $\bar{X}$.

We define a metric from $\tilde{\mathcal{K}} = \pm \log \pm \tilde{K}$, where $\pm$ is the sign of $\tilde{K}$. In comparison to section 5.1 we have now put $\kappa = 1$ and $a = 1$, and there we assumed that $\tilde{K} = -N$ is negative.

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} \tilde{\mathcal{K}} = \partial_I \partial_{\bar{J}} \tilde{\mathcal{K}} = \pm \frac{1}{\tilde{K}} \partial_I \partial_{\bar{J}} \tilde{K} \mp \frac{1}{\tilde{K}^2} \partial_I \tilde{K} \partial_{\bar{J}} \tilde{K} \,. \tag{6.95}$$

This metric has zero modes X^I. Orthogonal to it, it is non-degenerate if $\tilde{g}$ is non-degenerate and it has the signature of $\tilde{g}$ orthogonal to X^I. Note that the remaining signature of $\tilde{g}$ is the sign of $\tilde{K}$.

We can then define variables Y and z^α that form together an alternative basis for the X^I as in (5.12). The metric on the manifold $\mathcal{M}$ is taken to be

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z}), \qquad K(z, \bar{z}) = \pm \log \pm \frac{K}{Y\bar{Y}} \,, \qquad \pm = \operatorname{sign} \mathcal{K}. \tag{6.96}$$

The Kähler class of this manifold is then equal to the Chern class of the line bundle build from this manifold by the action of H (i.e. locally $\mathcal{M} \times \mathbb{C}^\times$). The Chern class of a line bundle is integer, and therefore the Kähler form is integer. Such manifolds are called *Kähler-Hodge manifolds* in the literature. What we have proven here is that a projective Kähler manifold is a Kähler-Hodge manifold. See [59] for more detailed explanations.

What we have explained here is applicable to any Kähler manifold with closed homothetic Killing vector. When one starts with a rigid special Kähler manifold, the resulting projective Kähler manifold is the special Kähler manifold that we obtain as target manifold in $N = 2$ supergravity in $d = 4$ coupled to vector multiplets.

6.4 Electric and magnetic charges and the attractor phenomenon

The attractor mechanism in special geometry [92, 93] is the fact that black hole type solutions are such that the scalars take at the horizon of charged black holes fixed values that are determined by these charges irrespective of the initial condition, i.e. the values of the scalars at infinite distance from the black hole. We restrict here to 4 dimensional theories.[25]

6.4.1 The spacetime ansatz and an effective action

A static spacetime metric[26] has the general form

$$\begin{array}{l}\mathrm{d}s^2 = -\mathrm{e}^{2U}\mathrm{d}t\mathrm{d}t + \mathrm{e}^{-2U}\gamma_{mn}\mathrm{d}x^m\mathrm{d}x^n, \qquad \text{i.e. } g_{00} = -\mathrm{e}^{2U}, \quad g_{mn} = \mathrm{e}^{-2U}\gamma_{mn}, \\ \partial_t U = \partial_t\gamma_{mn} = 0. \end{array} \tag{6.97}$$

This leads to

$$S_{\text{Einstein}} = \int \mathrm{d}^4x \tfrac12\sqrt{g}R(g) = \int \mathrm{d}^4x\sqrt{\gamma}\left(\tfrac12 R(\gamma) - \partial_m U\partial^m U + D_m\partial^m U\right). \tag{6.98}$$

The last term can be omitted as it is a total space derivative. Indeed, here γ_{mn} is used to raise and lower indices, and to define the covariant derivative D_m.

Note, however, that the field equation obtained from (6.98) and other matter terms is not sufficient to satisfy all Einstein equations. Let us consider this in detail. The Einstein tensor is

$$G_{\mu\nu} = 2(\sqrt{g})^{-1}\frac{\delta S_{\text{Einstein}}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \tfrac12 g_{\mu\nu}R\,. \tag{6.99}$$

For the static metric, one obtains

$$\begin{aligned} G_{00} &= \mathrm{e}^{4U}\left(\tfrac12 R(\gamma) + 2D_m\partial^m U - \partial^m U\partial_m U\right), \\ G_{mn} &= \left(\gamma_{mn}\gamma^{rs} - 2\delta_m^r\delta_n^s\right)\left(-\tfrac12 R_{rs}(\gamma) + \partial_r U\partial_s U\right). \end{aligned} \tag{6.100}$$

We consider now only the bosonic sector. The Einstein equations are

$$G_{\mu\nu} = T_{\mu\nu}\,, \tag{6.101}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. We split it in the scalar part and the spin 1 part:

$$\begin{aligned} T_{\mu\nu} &= T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}, \\ & T^{(0)}_{\mu\nu} = -2(\sqrt{g})^{-1}\frac{\delta S^{(0)}}{\delta g^{\mu\nu}}, \qquad T^{(1)}_{\mu\nu} = -2(\sqrt{g})^{-1}\frac{\delta S^{(1)}}{\delta g^{\mu\nu}}\,. \end{aligned} \tag{6.102}$$

[25] The attractor phenomenon in 5 (and 6) dimensional $N = 2$ theories has been considered in [94, 95, 96]

[26] 'Static' means that it admits a global, nowhere zero, timelike hypersurface orthogonal Killing vector field. A generalization are the 'stationary' spacetimes, which admit a global, nowhere zero timelike Killing vector field. In that case the components g_{0m} could be nonzero. For simplicity we look here to the static spacetimes.

Schematically, we have

$$S^{(0)}=-\int \mathrm{d}^4x\sqrt{g}g^{\mu\nu}\partial_\mu z\partial_\nu\bar{z}\,. \tag{6.103}$$

Of course there are many scalars in the Kähler manifold, but generalizing to z^α with a Kähler metric $g_{\alpha\bar\beta}$ is obvious in this section, and we will thus omit it for simplicity of the formulae.

The scalar energy-momentum tensor is

$$T^{(0)}_{\mu\nu}=-g_{\mu\nu}g^{\rho\sigma}\partial_\rho z\partial_\sigma\bar{z}+2\partial_\mu z\partial_\nu\bar{z}\,. \tag{6.104}$$

For our metric ansatz, this gives

$$T^{(0)}_{00}=\mathrm{e}^{4U}\gamma^{mn}\partial_m z\partial_n\bar{z}\,,\qquad T^{(0)}_{mn}=\left(-\gamma_{mn}\gamma^{rs}+2\delta^r_{(m}\delta^s_{n)}\right)\partial_r z\partial_s\bar{z}\,. \tag{6.105}$$

In the spin-1 action,

$$S^{(1)}=\int \mathrm{d}^4x\left[\tfrac14\sqrt{g}(\mathrm{Im}\,\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J}-\tfrac18(\mathrm{Re}\,\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\right]\,, \tag{6.106}$$

the metric appears only in the term with $\mathrm{Im}\,\mathcal{N}$. This leads to

$$T^{(1)}_{\mu\nu}=-\,\mathrm{Im}\,\mathcal{N}_{IJ}\left(-\tfrac14 g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}+F^I_{\mu\rho}F^J_{\nu\sigma}g^{\rho\sigma}\right)\,. \tag{6.107}$$

If we now use the spacetime metric (6.97), the nonzero terms are

$$\begin{aligned}T^{(1)}_{00}&=-\,\mathrm{Im}\,\mathcal{N}_{IJ}\left(\tfrac12\mathrm{e}^{2U}F^I_{0m}\gamma^{mn}F^J_{0n}+\tfrac14\mathrm{e}^{6U}F^I_{mn}\gamma^{mp}\gamma^{nq}F^J_{pq}\right)\\ T^{(1)}_{mn}&=-\,\mathrm{Im}\,\mathcal{N}_{IJ}\left(\tfrac12\mathrm{e}^{-2U}\gamma_{mn}F^I_{0p}\gamma^{pq}F^J_{0q}-\tfrac14\mathrm{e}^{2U}\gamma_{mn}F^I_{pq}\gamma^{pp'}\gamma^{qq'}F^J_{p'q'}\right.\\ &\qquad\left.-\mathrm{e}^{-2U}F^I_{0m}F^J_{0n}+\mathrm{e}^{2U}F^I_{mp}\gamma^{pq}F^J_{nq}\right)\end{aligned} \tag{6.108}$$

We can use the magnetic vectors

$$F^I_m=\tfrac12\gamma_{mn}(\sqrt{\gamma})^{-1}\varepsilon^{npq}F^I_{pq}, \tag{6.109}$$

which has the useful consequences

$$\begin{aligned}&F^I_{mn}=\sqrt{\gamma}\varepsilon_{mnp}\gamma^{pq}F^I_q,\\ &F^I_{mn}\gamma^{mp}\gamma^{nq}F^J_{pq}=2F^I_r\gamma^{rs}F^J_s,\qquad F^I_{mp}\gamma^{pq}F^J_{nq}=\gamma_{mn}F^I_r\gamma^{rs}F^J_s-F^I_nF^J_m.\end{aligned} \tag{6.110}$$

Using this, we can again write the energy-momentum tensor in a similar form as for the gravity field and for the scalars: We find

$$T^{(1)}_{00}=\mathrm{e}^{6U}\gamma^{mn}V_{mn}\,,\qquad T^{(1)}_{mn}=\mathrm{e}^{2U}\left(\gamma_{mn}\gamma^{rs}-2\delta^r_m\delta^s_n\right)V_{rs}\,, \tag{6.111}$$

where

$$V_{mn}=-\tfrac12\,\mathrm{Im}\,\mathcal{N}_{IJ}\left(\mathrm{e}^{-4U}F^I_{0m}F^J_{0n}+F^I_mF^J_n\right)\,. \tag{6.112}$$

The Einstein equations thus reduce to the following two equations

$$-\tfrac{1}{2}R_{mn}(\gamma) + \partial_m U\,\partial_n U + \partial_{(m} z \partial_{n)}\bar{z} - \mathrm{e}^{2U} V_{mn} = 0\,, \qquad (6.113)$$

$$D_m \partial^m U - \mathrm{e}^{2U}\gamma^{mn} V_{mn} = 0\,. \qquad (6.114)$$

We will propose the following effective action

$$S_{\text{eff}} = \int \mathrm{d}^3x \sqrt{\gamma}\left[-\partial_m U\,\gamma^{mn}\partial_n U - \partial_m z\,\gamma^{mn}\partial_n \bar{z} - \mathrm{e}^{2U}\gamma^{mn}V_{mn}\right]\,. \qquad (6.115)$$

The field equation of this action for U is (6.114) if we keep V_{mn} fixed during the variation. Of course, as U does not appear in the scalar part of the action, the latter is not determined by this requirement. We will prove that the field equations for the scalars can also be derived from this action for a specific parametrization of V_{mn}. This will be clarified in section 6.4.2. Only then it will be clear how to use this effective action.

The action (6.115) has to be supplemented with the extra constraint (6.113), which is not derivable from the effective action.

6.4.2 Maxwell equations and the black hole potential

(6.112) is expressed in components of the field strengths $F^I_{\mu\nu}$. However, we can write it in terms of the symplectic vectors of field strengths and field equations. To do so, we use the real form of (6.1). Using (A.12), this is

$$G_{I\mu\nu} = \mathrm{Re}\,\mathcal{N}_{IJ}F^J_{\mu\nu} + \tfrac{1}{2}\,\mathrm{Im}\,\mathcal{N}_{IJ} g_{\mu\mu'} g_{\nu\nu'} (\sqrt{g})^{-1}\varepsilon^{\mu'\nu'\rho\sigma}F^J_{\rho\sigma}\,. \qquad (6.116)$$

If we denote the 3-dimensional duals as a generalization of (6.109):

$$F^I_m = \tfrac{1}{2}\gamma_{mn}(\sqrt{\gamma})^{-1}\varepsilon^{npq}F^I_{pq}, \qquad G_{Im} = \tfrac{1}{2}\gamma_{mn}(\sqrt{\gamma})^{-1}\varepsilon^{npq}G_{Ipq}, \qquad (6.117)$$

we obtain from (6.116) and $\varepsilon^{0npq} = -\varepsilon^{npq}$ (as we use $\varepsilon_{0123} = 1$, see (A.6))

$$\begin{pmatrix} F^I_{0m} \\ G_{I0m} \end{pmatrix} = -\mathrm{e}^{2U}\mathcal{M}\Omega \begin{pmatrix} F^J_m \\ G_{Jm} \end{pmatrix}, \qquad (6.118)$$

where $\mathcal{M}$ was given in (6.74) and Ω is the symplectic metric (6.7). These matrices contain indices I and J at appropriate positions automatically for (6.118) to make sense. This leads to

$$V_{mn} = \frac{1}{2}\begin{pmatrix} F^I_m & G_{Im} \end{pmatrix}\Omega\mathcal{M}\Omega \begin{pmatrix} F^J_n \\ G_{Jn} \end{pmatrix}\,. \qquad (6.119)$$

Explicitly we have

$$\Omega\mathcal{M}\Omega = \begin{pmatrix} -I - RI^{-1}R & RI^{-1} \\ I^{-1}R & -I^{-1} \end{pmatrix}, \qquad R = \mathrm{Re}\,\mathcal{N}\,, \quad I = \mathrm{Im}\,\mathcal{N}\,. \qquad (6.120)$$

U does not appear in this expression for V_{mn}. This implies that if we consider V_{mn} as a function of F^I_m, G_{Im} and the scalars implicitly present in (6.120) and insert it as such in the effective action (6.115) then it is still valid that this action generates the right field equation for U. We now check that in this way it also generates the same scalar field equations as those obtained from the original action $S^{(0)}$ and $S^{(1)}$, where the vector fields W^I_μ where the other independent variables. Hence these field equations that should be reproduced are

$$0 = \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \bar{z} + \tfrac{1}{4}\sqrt{g}\partial_z (\mathrm{Im}\,\mathcal{N}_{IJ}) F^I_{\mu\nu} F^{\mu\nu J} - \tfrac{1}{8}\partial_z (\mathrm{Re}\,\mathcal{N}_{IJ}) \varepsilon^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^J_{\rho\sigma}\,. \tag{6.121}$$

Specifying the metric (6.97) and the expressions for the field strengths in terms of F^I_m and G_{Im}, this becomes

$$0 = \partial_m \sqrt{\gamma}\gamma^{mn} \partial_n \bar{z} + \frac{1}{2}\sqrt{\gamma}\mathrm{e}^{2U}\gamma^{mn} \begin{pmatrix} F^I_m & G_{mI} \end{pmatrix} \Omega \partial_z \mathcal{M} \Omega \begin{pmatrix} F^J_n \\ G_{nJ} \end{pmatrix}\,, \tag{6.122}$$

where the indices I and J appear again in appropriate positions on the submatrices of $\Omega\partial_z\mathcal{M}\Omega$. The latter is indeed the field equation obtained from the effective action

$$S_{\mathrm{eff}}(U,z) = \int \mathrm{d}^3 x \sqrt{\gamma}\gamma^{mn} \left[-\partial_m U\, \partial_n U - \partial_m z\, \partial_n \bar{z} - \frac{1}{2}\mathrm{e}^{2U} \begin{pmatrix} F^I_m & G_{Im} \end{pmatrix} \Omega \mathcal{M} \Omega \begin{pmatrix} F^J_n \\ G_{Jn} \end{pmatrix} \right]. \tag{6.123}$$

The (U,z) in the left-hand side indicates that it should be considered as an effective action for varying with respect to these variables, while γ^{mn}, F^I_m and G_{Im} should be considered as background. We saw already that the field equations of the original action for γ^{mn} lead to the constraint (6.114). We will now check what the field equations of the vector sector impose.

The field equations from (6.106) with independent vectors W^I_μ are equivalent to the field equations and Bianchi identities

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu \begin{pmatrix} F^I_{\rho\sigma} \\ G_{I\rho\sigma} \end{pmatrix} = 0\,. \tag{6.124}$$

Using our preferred variables, this gives

$$\partial_m \sqrt{\gamma}\gamma^{mn} \begin{pmatrix} F^I_n \\ G_{In} \end{pmatrix} = 0, \qquad \partial_{[m} \mathrm{e}^{2U}\mathcal{M}\Omega \begin{pmatrix} F^I_{n]} \\ G_{In]} \end{pmatrix} = 0. \tag{6.125}$$

One way of solving these equations is to put

$$\begin{aligned} &F^I_m = \partial_m H^I\,, \qquad G_{Im} = \partial_m H_I\,, \\ &\partial_m \sqrt{\gamma}\gamma^{mn}\partial_n H^I = \partial_m \sqrt{\gamma}\gamma^{mn}\partial_n H_I = 0\,. \end{aligned} \tag{6.126}$$

We remain then with Bianchi identities of the form

$$\left(\partial_{[m}\mathrm{e}^{2U}\mathcal{M}\right)\Omega\partial_{n]} \begin{pmatrix} H^I \\ H_I \end{pmatrix} = 0\,, \tag{6.127}$$

which can be solved by assuming that all functions (U, the scalars and the harmonic H^I and H_I) depend only on one coordinate such that the ∂_m and ∂_n for $m \neq n$ in the above equation cannot both be nonvanishing. We denote this one coordinate as τ. Thus $U(\tau)$, $z(\tau)$, $H^I(\tau)$ and $H_I(\tau)$.

A convenient metric is e.g. [97]

$$\gamma_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{c^4}{\sinh^4 c\tau}\mathrm{d}\tau^2 + \frac{c^2}{\sinh^2 c\tau}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{6.128}$$

Details on this parametrization are given in an appendix of [98]. This parametrization has the property $\sqrt{\gamma}\gamma^{\tau\tau} = \sin\theta$, which will be useful.

Harmonic means now just $H'' = 0$, where a prime is now a derivative w.r.t. τ. So we can take

$$H = \begin{pmatrix} H^I \\ H_I \end{pmatrix} = \Gamma\tau + h, \qquad h = \begin{pmatrix} h^I \\ h_I \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} p^I \\ q_I \end{pmatrix}. \tag{6.129}$$

We have here introduced the magnetic and electric charges in the symplectic vector Γ. We come back to their definition in section 6.4.3.

In this parametrization, the action gets the form (up to a constant)

$$S_{\text{eff}} = U'^2 + \mathrm{e}^{2U}V_{\text{BH}} - z'\bar{z}'\,, \tag{6.130}$$

where the 'black hole potential is

$$V_{\text{BH}} = V_{\tau\tau} = \tfrac{1}{2}\Gamma^T\Omega\mathcal{M}\Omega\Gamma. \tag{6.131}$$

The Einstein equations lead to 2 independent equations:

$$c^2 - U'^2 - z'\bar{z}' + \mathrm{e}^{2U}V_{\text{BH}} = 0. \qquad -U'' + \mathrm{e}^{2U}V_{\text{BH}} = 0\,. \tag{6.132}$$

The second one is the one that can be obtained from the effective action. The first one is an extra constraint.

6.4.3 Field strengths and charges

We consider field configurations with electric and/or magnetic charges in 4 dimensions. This means that there are 2-cycles S^2 surrounding the sources such that

$$\int_{S^2} F^I_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 8\pi p^I, \qquad \int_{S^2} G_{I\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 8\pi q_I. \tag{6.133}$$

Exercise 6.4: Check that the solution that we gave above, leads indeed to the identification of the charges here and in (6.129)

There is also the field strength that occurs in the gravitino transformation, see e.g. (3.70), which is the value of the auxiliary field $T_{\mu\nu}$ of the Weyl multiplet. When we restrict now to the bosonic part in its value, determined in (5.63), we obtain

$$\begin{aligned} T^-_{\mu\nu} &= 2T_I F^{-I}_{\mu\nu} \qquad (6.134)\\ T_I &= \frac{N_{IJ}\bar{X}^J}{\bar{X}^L N_{LM}\bar{X}^M} = -2\,\mathrm{Im}\,\mathcal{N}_{IJ}X^J = \mathrm{i}\left(F_I - \bar{\mathcal{N}}_{IJ}X^J\right), \qquad \bar{X}^I T_I = N = 1, \end{aligned}$$

where use has been made of (5.39).

The integral of this quantity gives (we assume here and below that the $F_I(z)$ and $X^I(z)$ are sufficiently constant in the integration region such that they can be taken in and out of the integral) [99]

$$Z \equiv \frac{\mathrm{i}}{16\pi}\int_{S^2} T^-_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = \frac{1}{8\pi}\int_{S^2}\left(X^I G^-_{\mu\nu} - F_I F^{-I}_{\mu\nu}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$
$$= \frac{1}{8\pi}\int_{S^2}\left(X^I G_{I\mu\nu} - F_I F^I_{\mu\nu}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = X^I q_I - F_I p^I. \qquad (6.135)$$

Note that the combination with the selfdual field strengths vanishes due to $F_I = \mathcal{N}_{IJ}X^J$. The object Z is called the central charge, because its value appears in the commutator of two supersymmetries, as can be seen from (3.73), (3.74).

However, when we take the covariant derivatives of the final expression, then we have to use $\mathcal{D}_\alpha F_I = \bar{\mathcal{N}}_{IJ}\mathcal{D}_\alpha X^J$, and therefore only the selfdual parts remain. This gives thus

$$\mathcal{D}_\alpha Z = \mathcal{D}_\alpha X^I q_I - \mathcal{D}_\alpha F_I p^I \qquad (6.136)$$
$$= \frac{1}{8\pi}\int_{S^2}\left(\mathcal{D}_\alpha X^I G^+_{I\mu\nu} - \mathcal{D}_\alpha F_I F^{+I}_{\mu\nu}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$
$$= \frac{2\mathrm{i}}{8\pi}\int_{S^2}\mathcal{D}_\alpha X^I\,\mathrm{Im}\,\mathcal{N}_{IJ}F^{+I}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu.$$

The latter quantities are the objects that appear also in the transformation laws of the physical gauginos. Indeed, the fermions of the conformal multiplets transform according to (4.10) in quantities $\mathcal{F}_{ab}$, whose bosonic part is now

$$\mathcal{F}^{-I}_{\mu\nu} = \left(\delta^I_J - \bar{X}^I T_J\right)F^{-J}_{\mu\nu}. \qquad (6.137)$$

The physical fermions are the ones that satisfy the S-gauge condition (5.68), which means that they vanish under projection with T_I. We find indeed $T_I\mathcal{F}^{-I}_{\mu\nu} = 0$.

In a symplectic notation, we can define the vector

$$\Gamma = \begin{pmatrix} p^I \\ q_I \end{pmatrix}. \qquad (6.138)$$

Then we have $Z = \langle V,\Gamma\rangle = V^T\Omega\Gamma$ and $\mathcal{D}_\alpha Z = \langle U_\alpha,\Gamma\rangle$. This leads immediately to a simple expression for the 'black hole potential' [100, 97]

$$V_{\rm BH} \equiv Z\bar{Z} + \mathcal{D}_\alpha Z g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar{Z} = \tfrac{1}{2}\Gamma^T\Omega\mathcal{M}\Omega\Gamma, \qquad (6.139)$$

which can be derived easily from (6.73), but was already found in [91]. Similarly one derives using the same identity

$$V\bar{Z} + \bar{U}_{\bar\beta}g^{\bar\beta\alpha}\mathcal{D}_\alpha Z = -\tfrac{1}{2}\left(\mathcal{M}\Omega + \mathrm{i}\mathbb{1}\right)\Gamma. \qquad (6.140)$$

6.4.4 Attractors

The attractor solution [92, 93, 100] is the solution near the horizon. This is thus the large τ behaviour of what we mentioned above. In that case supersymmetry is preserved, which is expressed as $\mathcal{D}_\alpha Z = 0$. This extremizes the black hole potential. So it is consistent with constant z^α as solution of the field equation for the scalars. In this case (6.140) simplifies. Taking the imaginary part, gives

$$-2\,\mathrm{Im}(V\bar{Z}) = \Gamma. \tag{6.141}$$

These are the attractor equations. The BH potential reduces to

$$V_{\mathrm{BH,BPS}} = |Z|^2. \tag{6.142}$$

Then we determine U by the constraint

$$\dot{U}^2 = \mathrm{e}^{2U} V_{\mathrm{BH,BPS}}, \qquad \text{i.e.} \qquad \dot{U} = \pm \mathrm{e}^{U} \tag{6.143}$$

The $V_{\mathrm{BH,BPS}}$ being constant, this automatically implies the other field equation

$$\ddot{U} = \mathrm{e}^{2U} V_{\mathrm{BH,BPS}}. \tag{6.144}$$

To finish the solution, we may write

$$\mathrm{e}^{-U} = \mp |Z| \tau + \text{constant}. \tag{6.145}$$

6.5 Gauge transformations as isometries

I now consider in more detail the gauge group gauged by the vectors of the vector multiplets in 4 dimensions. As the scalars are by supersymmetry connected to the gauge vectors, the gauge transformations of the X^I are fixed to be in the adjoint representation. I will indicate the parameters by α^I. In the context of the tensor calculus we thus have

$$\delta_G X^I = g\alpha^J X^K f_{KJ}{}^I. \tag{6.146}$$

We will now first consider general isometries of the metric, possibly not gauged. We will indicate these by an index Λ. Such isometries act on the scalars defining the 'Killing vector' k_Λ^α:

$$\delta_G z^\alpha = -g\alpha^\Lambda k_\Lambda^\alpha(z). \tag{6.147}$$

We assume here already that the Killing vectors are holomorphic such that the complex structure is preserved. is not necessary . Also the compensator field can transform. To respect the dilatational structure, the form should be

$$\delta_G Y = -g\alpha^\Lambda Y r_\Lambda(z), \tag{6.148}$$

for some holomorphic function $r_\Lambda(z)$. Notice that this transformation does not leave the gauge choice (5.30) invariant, as we have $\delta_{\Lambda_A} Y = -\mathrm{i}\Lambda_A Y$. Hence, such a gauge choice is only invariant under a linear combination of these G-transformations and of the original conformal U(1). In other words, there is a 'decomposition law' that says that for the resulting transformations after imposing this gauge choice are those with

$$\Lambda_A = \tfrac{1}{2}\mathrm{i}g\alpha^\Lambda \left[r_\Lambda(z) - \bar{r}_\Lambda(\bar{z}) \right]. \tag{6.149}$$

The Kähler potential is in a U(1)-invariant way defined as in (5.29), and thus we obtain

$$\delta_G\mathcal{K}(z,\bar{z}) = -g\alpha^\Lambda \left[k_\Lambda^\alpha(z)\partial_\alpha\mathcal{K}(z,\bar{z}) + k_\Lambda^{\bar{\alpha}}(\bar{z})\partial_{\bar{\alpha}}\mathcal{K}(z,\bar{z}) \right] = -g\alpha^\Lambda \left[r_\Lambda(z) + \bar{r}_\Lambda(\bar{z}) \right]. \tag{6.150}$$

We see here that the possibility of (6.148) corresponds to the fact that the isometries do not necessarily leave the Kähler potential invariant. It is sufficient that the Kähler potential transforms with a Kähler transformation (5.35) depending on a holomorphic function.

This equation can be expressed as the reality condition of a function

$$\begin{aligned} P_\Lambda^0 &\equiv \mathrm{i}k_\Lambda^\alpha(z)\partial_\alpha\mathcal{K}(z,\bar{z}) - \mathrm{i}r_\Lambda(z) \\ &= -\mathrm{i}k_\Lambda^{\bar{\alpha}}(\bar{z})\partial_{\bar{\alpha}}\mathcal{K}(z,\bar{z}) + \mathrm{i}\bar{r}_\Lambda(\bar{z}) \\ &= \tfrac{1}{2}\mathrm{i}\left[k_\Lambda^\alpha(z)\partial_\alpha\mathcal{K}(z,\bar{z}) - k_\Lambda^{\bar{\alpha}}(\bar{z})\partial_{\bar{\alpha}}\mathcal{K}(z,\bar{z}) - r_\Lambda(z) + \bar{r}_\Lambda(\bar{z}) \right]. \end{aligned} \tag{6.151}$$

This is the so-called'moment map' function P_Λ^0 (the upper index 0 is useless here, but is introduced in the context of other triple moment maps that appear for quaternionic-Kähler manifolds).

The Killing vectors can be obtained from the moment map, as one can easily check:

$$\partial_\alpha P_\Lambda^0(z,\bar{z}) = -\mathrm{i}k_\Lambda^{\bar{\alpha}}(z,\bar{z})g_{\alpha\bar{\alpha}}(z,\bar{z}), \qquad \partial_{\bar{\alpha}} P_\Lambda^0(z,\bar{z}) = \mathrm{i}k_\Lambda^\alpha(z,\bar{z})g_{\bar{\alpha}\alpha}(z,\bar{z}). \tag{6.152}$$

In the context of the symplectic sections, which we introduced in (6.36), one can check that

$$\begin{aligned} g\alpha^\Lambda k_\Lambda^\alpha \mathcal{D}_\alpha v(z) &= g\alpha^\Lambda k_\Lambda^\alpha \left[\partial_\alpha v(z) + v(z)\partial_\alpha\mathcal{K} \right] \\ &= -\delta_G v(z) + g\alpha^\Lambda \left(r_\Lambda - \mathrm{i}P_\Lambda^0 \right) v(z). \end{aligned} \tag{6.153}$$

One can then use (6.28) and its derivative

$$\langle \mathcal{D}_\alpha v, \bar{v} \rangle = 0. \tag{6.154}$$

to obtain

$$g\mathrm{e}^{-\mathcal{K}}\alpha^\Lambda \left[P_\Lambda^0 + \mathrm{i}r_\Lambda \right] = \langle \delta_G v, \bar{v} \rangle. \tag{6.155}$$

Subtracting the complex conjugate shows that

$$\delta_G \langle v, \bar{v} \rangle = \mathrm{i}g\mathrm{e}^{-\mathcal{K}}\alpha^\Lambda \left[r_\Lambda + \bar{r}_\Lambda \right]. \tag{6.156}$$

Thus the r_Λ part does not preserve symplectic products and does not belong to a symplectic transformation. The invariance group of the special Kähler manifolds contains the symplectic transformations and Kähler transformations. We should in general thus have

$$\delta_G v = g\alpha^\Lambda \left[T_\Lambda v + r_\Lambda v \right], \tag{6.157}$$

The matrix T_Λ should be an infinitesimal symplectic transformation while r_Λ is a holomorphic scalar function, for which we have taken already the normalization that is clear from the previous equations. We thus obtain

$$P^0_\Lambda = \mathrm{e}^{\mathcal{K}}\langle T_\Lambda v, \bar{v}\rangle. \tag{6.158}$$

In terms of the $V(z,\bar{z})$ (using the U(1) gauge fixing such that (5.31) is satisfied) we have

$$\begin{aligned} \delta_G V_{\mathrm{gf}} &= g\alpha^\Lambda\left[-k^\alpha_\Lambda \mathcal{D}_\alpha + \tfrac{1}{2}r_\Lambda - \tfrac{1}{2}\bar{r}_\Lambda - \mathrm{i}P^0_\Lambda\right]V \\ &= g\alpha^\Lambda\left[T_\Lambda V + \tfrac{1}{2}(r_\Lambda - \bar{r}_\Lambda)V\right], \\ P^0_\Lambda &= \langle T_\Lambda V, \bar{V}\rangle. \end{aligned} \tag{6.159}$$

The subscript gf indicates that this is after gauge fixing. We know the effect of this gauge fixing: it is the compensating U(1) transformation (6.149). Thus, the r-dependent part can be seen as the result of this compensating transformation. The transformation without the U(1) part is thus

$$\delta_G V = -g\alpha^\Lambda\left[k^\alpha_\Lambda \mathcal{D}_\alpha + \mathrm{i}P^0_\Lambda\right]V = g\alpha^\Lambda T_\Lambda V. \tag{6.160}$$

Exercise 6.5: Check the following relation that appears in the potential:

$$k^\alpha_\Lambda g_{\alpha\bar{\beta}} k^{\bar{\beta}}_\Sigma = \mathrm{i}k^\alpha_\Lambda \langle U_\alpha, \bar{U}_{\bar{\beta}}\rangle k^{\bar{\beta}}_\Sigma = \mathrm{i}\langle T_\Lambda V, T_\Sigma \bar{V}\rangle + P^0_\Lambda P^0_\Sigma. \tag{6.161}$$

When there are different transformations denoted by the index Λ, they satisfy an algebra. Thus, the commutator of two transformations (6.147) should give another one, or

$$k^\beta_\Lambda \partial_\beta k^\alpha_\Sigma - k^\beta_\Sigma \partial_\beta k^\alpha_\Lambda = -f_{\Lambda\Sigma}{}^\Gamma k^\alpha_\Gamma \tag{6.162}$$

Imposing the same relation for (6.148) leads to the condition

$$k^\alpha_\Lambda \partial_\alpha r_\Sigma - k^\alpha_\Sigma \partial_\alpha r_\Lambda = -f_{\Lambda\Sigma}{}^\Gamma r_\Gamma. \tag{6.163}$$

Using (6.150), we obtain

$$2k^\alpha_{[\Lambda} k^{\bar{\beta}}_{\Sigma]} g_{\alpha\bar{\beta}} + \mathrm{i}f_{\Lambda\Sigma}{}^\Gamma P^0_\Gamma = 0. \tag{6.164}$$

This condition is often called the 'equivariance' condition.

Up till here, this holds for any isometry indicated by Λ. Now we specify to the gauged isometries, a subgroup of the full set of isometries. The index Λ is then fixed to the range I as we need vectors to gauge them. For the reasons mentioned in the beginning of this section, the transformation of X^I is fixed as in (6.146). The transformations on the symplectic sections should be embedded in the symplectic group or in the Kähler transformations. As it was mentioned after (6.159), the r-dependent parts in $\delta_G V$ can be seen as originating from the compensating U(1) transformations. Hence, the transformation (6.146) should be identified with the symplectic part (6.160). The symplectic transformations should be infinitesimal transformations of the form of (6.5), with moreover $B = 0$ to be symmetries of the action. This means that $A^I{}_J = \delta^I_J - \alpha^K f_{KJ}{}^I$ and then (6.5) determine that

$$T_K V = \begin{pmatrix} -f_{KJ}{}^I & 0 \\ C_{K,IJ} & f_{KI}{}^J \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix}, \tag{6.165}$$

where $C_{K,IJ}$ is symmetric in the last two indices. In order that these matrices satisfy the algebra, we have to require that

$$f_{KL}{}^M C_{M,IJ} = 2 f_{J[K}{}^M C_{L],IM} + 2 f_{I[K}{}^M C_{L],JM}. \tag{6.166}$$

Note that the transformation of $X^I F_I$ is

$$\delta_G(X^I F_I) = \alpha^K C_{K,IJ} X^I X^J. \tag{6.167}$$

If the X^I are independent $X^I F_I = F(X)$, and the latter thus transforms in a real quadratic function, which does not contribute to the action.

The gauge transformation of X^I and F_I can also be obtained from (6.160):

$$\begin{aligned} \delta_G X^I &= -g\alpha^J \left(k_J^\alpha \mathcal{D}_\alpha + \mathrm{i} P_J^0\right) X^I \\ \delta_G F_I &= -g\alpha^J \left(k_J^\alpha \mathcal{D}_\alpha + \mathrm{i} P_J^0\right) F_I. \end{aligned} \tag{6.168}$$

It is clear from (6.165) that $\delta_G X^I$ vanishes if we replace the parameter α^I by X^I. Therefore we find

$$X^J k_J^\alpha \mathcal{D}_\alpha X^I + \mathrm{i} P_J^0 X^J X^I = 0. \tag{6.169}$$

The same operation on the expression for $\delta_G F_I$ from (6.165) and from (6.168) gives two expressions that have to be identified:

$$X^K C_{K,IJ} X^J + X^K f_{KI}{}^J F_J + X^J k_J^\alpha \mathcal{D}_\alpha F_I + \mathrm{i} P_J^0 X^J F_I = 0. \tag{6.170}$$

Multiplying this with X^I and using (6.169) gives

$$X^I X^J C_{J,IK} X^K = -X^I X^J k_J^\alpha \mathcal{D}_\alpha F_I + X^J k_J^\alpha F_I \mathcal{D}_\alpha X^I = X^J k_J^\alpha \langle V, \mathcal{D}_\alpha V \rangle = 0. \tag{6.171}$$

This leads to the equation found in the context of a prepotential in [56] that the completely symmetric part of $C_{I,JK}$ should vanish if the X^I are independent:

$$C_{(I,JK)} X^I X^J X^K = 0. \tag{6.172}$$

6.6 Realizations in Calabi–Yau moduli spaces.

The realizations of special Kähler geometry that are mostly studied in physics these days are those of Calabi–Yau 3-folds.

To obtain local special Kähler manifolds, one considers the moduli space of Calabi–Yau 3-folds (see the lectures of P. Candelas XXX in this school). In this case the Hodge diamond of the manifold is

$$\begin{array}{ccccccccc} & & & & h^{00}=1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{11}=m & & 0 & & \\ h^{30}=1 & & & h^{21}=n & & h^{12}=n & & & h^{03}=1 \\ & & 0 & & h^{22}=m & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & h^{33}=1 & & & & \end{array}$$

These manifolds have $h^{21}=n$ complex structure moduli, which play the role of the variables z^α of the previous section. There are $2(n+1)$ 3-cycles c_Λ, with intersection matrix $Q_{\Lambda\Sigma} = c_\Lambda \cap c_\Sigma$. The canonical form is obtained with so-called A and B cycles, and then Q takes the form of Ω in (6.7). Symplectic vectors are identified again as vectors of integrals over the $2(n+1)$ 3-cycles:

$$v = \int_{c_\Lambda} \Omega^{(3,0)}\,, \qquad \mathcal{D}_\alpha v = \int_{c_\Lambda} \Omega_\alpha^{(2,1)}\,. \tag{6.173}$$

$\Omega^{(3,0)}$ is the unique $(3,0)$ form that characterizes the Calabi–Yau manifold. $\Omega_\alpha^{(2,1)}$ is a basis of the $(2,1)$ forms, determined by the choice of basis for z^α. That these moduli spaces give rise to special Kähler geometry became clear in [101, 102, 103, 104, 105, 78]. Details on the relation between the geometric quantities and the fundamentals of special Kähler geometry have been discussed in [106, 82].

The defining equations of special Kähler geometry are automatically satisfied. E.g. one can easily see how the crucial equation (6.49) is realized:

$$\begin{aligned} \langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v\rangle &= \int_{c_\Lambda} \Omega_{(\alpha)}^{(2,1)} \cdot Q^{\Lambda\Sigma} \cdot \int_{c_\Sigma} \Omega_{(\beta)}^{(2,1)} \\ &= \int_{CY} \Omega_{(\alpha)}^{(2,1)} \wedge \Omega_{(\beta)}^{(2,1)} = 0\,. \end{aligned} \tag{6.174}$$

The symplectic transformations correspond now to changes of the basis of the cycles used to construct the symplectic vectors. The statement that a formulation with a prepotential can always be obtained in special Kähler geometry by using a symplectic transformation, can now be translated to the statement that the geometry can be obtained from a prepotential for some choice of cycles. It is not yet clear whether the moduli spaces of Calabi–Yau manifolds have any parametrization without the existence of a prepotential. So far, this has never been seen, and it could be that in this subclass of special Kähler manifolds this phenomenon does not occur.

7 Action and main transformation laws from general considerations

The superconformal method gives a constructive way to obtain the full supergravity theories and understand their structure. Many aspects of the final result can be obtained also by some simple considerations on the supersymmetry algebra and invariance of the action. We consider here the case of $d = 4$.

The gravitino is normalized by that its supersymmetry transformation starts as a gauge field $\delta\psi_\mu = \partial_\mu \epsilon + \ldots$.

The leading (kinetic) terms of the action are

$$e^{-1}\mathcal{L}_{\rm kin} = \tfrac{1}{2}R - \bar\psi^i_\mu \gamma^{\mu\nu\rho}\mathcal{D}_\nu \psi_{\rho i}. \tag{7.1}$$

7.1 Vector multiplet and algebra

We use

$$\delta z^\alpha = \tfrac{1}{2}\bar\epsilon^i \lambda^\alpha_i , \qquad \delta\lambda^\alpha_i = \partial\!\!\!/ z^\alpha \epsilon_i . \tag{7.2}$$

Thus the supersymmetry algebra has leading term

$$[\delta_1, \delta_2] = \tfrac{1}{2}\bar\epsilon^i_2 \partial\!\!\!/ \epsilon_{1i} + \text{h.c.}. \tag{7.3}$$

The leading terms in the action are taken to be

$$\mathcal{L}_{\rm kin,special} = -g_{\alpha\bar\beta}\partial_\mu z^\alpha \partial^\mu \bar z^{\bar\beta} - \tfrac{1}{2} g_{\alpha\bar\beta}\bar\lambda^\alpha_i \partial\!\!\!/ \lambda^{\bar\beta i}. \tag{7.4}$$

This is invariant for constant g and constant ϵ, proving the normalization of the fermion kinetic term. Keeping ϵ spacetime dependent shows that there is a gravitino Noether term

$$\mathcal{L}_{\rm Noether,gaugino} = \tfrac{1}{2} g_{\alpha\bar\beta}\bar\lambda^\alpha_i \gamma^\mu (\partial\!\!\!/ \bar z^{\bar\beta})\psi^i_\mu + \text{h.c.}. \tag{7.5}$$

Vectors. We introduce vectors with kinetic terms

$$\mathcal{L}_1 = \tfrac{1}{2}\,{\rm Im}\left(\mathcal{N}_{IJ}(z)\, F^{+I}_{\mu\nu} F^{+\mu\nu\, J}\right) = \tfrac{1}{4}({\rm Im}\,\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J} - \tfrac{1}{8}({\rm Re}\,\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}. \tag{7.6}$$

We now add a central charge to the algebra

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \ldots + \delta_G\left(\varepsilon^{ij}\bar\epsilon_{2i}\epsilon_{1j}X + \text{h.c.}\right), \tag{7.7}$$

where δ_G is the gauge transformation, which is so far only relevant on the vector.

We first consider the λ terms in δW^I_μ such that this algebra is obtained, and the $\psi_\mu X$ terms follow also from the general rule of transformations of gauge fields:

$$\delta_Q(\epsilon)W^I_\mu = \tfrac{1}{2}(\mathcal{D}_\alpha X^I)\varepsilon^{ij}\bar\epsilon_i\gamma_\mu\lambda^\alpha_j + \tfrac{1}{2}(\mathcal{D}_{\bar\alpha}\bar X^I)\varepsilon_{ij}\bar\epsilon^i\gamma_\mu\lambda^{\bar\alpha j} + \varepsilon^{ij}\bar\epsilon_i\psi_{\mu j}X^I + \varepsilon_{ij}\bar\epsilon^i\psi^j_\mu \bar X^I. \tag{7.8}$$

The transformation of λ in (7.2) shows that the first term also leads to the commutator in (7.7).

Take e.g. only an imaginary $\mathcal{N}_{IJ}$, we can check the terms with constant scalars and $\partial\lambda F_{\mu\nu}$ terms. The kinetic terms of the vectors lead to

$$\delta\mathcal{L} = \tfrac{1}{2}\,{\rm Im}\,\mathcal{N}_{IJ}F^J_{\mu\nu}(\mathcal{D}_{\bar\alpha}\bar X^I)\varepsilon_{ij}\bar\epsilon^i\gamma_\nu\partial_\mu\lambda^{\bar\alpha j} + \text{h.c.}. \tag{7.9}$$

To cancel this term we need a transformation of the gaugino (to be used in the kinetic term of the gaugino)

$$\delta\lambda^\alpha_i = \nabla\!\!\!\!/\, z^\alpha\epsilon_i - \tfrac{1}{2}g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar X^I\,{\rm Im}\,\mathcal{N}_{IJ}F^{-J}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j. \tag{7.10}$$

7.2 Gravitino and gauge fields

Calculating the variation of the action proportional to $F_{\mu\nu}$ and $R_{\mu\nu}(Q)$ will lead to the transformation of the gravitino proportional to gauge fields. We use

$$\delta F^I_{\mu\nu} = \varepsilon^{ij}\bar{\epsilon}_i R_{\mu\nu j} X^I. \tag{7.11}$$

We consider all scalar fields constant, and as such drop terms like $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}R_{\rho\sigma}(Q)$. the most difficult part is that

$$\gamma_\mu \gamma^{ab}\gamma^{\mu\nu\rho}F^-_{ab} = -8F^-_{\nu\rho}, \tag{7.12}$$

where it has been used that F^- is anti-selfdual to combine two terms in the calculation. This leads to the result

$$\delta_Q(\epsilon)\psi^i_\mu = D_\mu(\omega)\epsilon^i + \tfrac{1}{4}\gamma\cdot F^{-I}\varepsilon^{ij}\gamma_\mu\epsilon_j(\operatorname{Im}\mathcal{N})_{IJ}X^J. \tag{7.13}$$

7.3 Hypermultiplet

The same algebra (7.3) is obtained from the (leading part of the) transformations

$$\begin{aligned}
\delta_Q(\epsilon)q^X &= -\mathrm{i}f^X_{iA}\bar{\epsilon}^i\zeta^A + \mathrm{i}f^{XiA}\bar{\epsilon}_i\zeta_A,\\
\delta_Q(\epsilon)\zeta^A &= \tfrac{1}{2}\mathrm{i}f^{iA}_X \partial\!\!\!/ q^X\epsilon_i,\\
\delta_Q(\epsilon)\zeta_A &= -\tfrac{1}{2}\mathrm{i}f_{XiA}\partial\!\!\!/ q^X\epsilon^i.
\end{aligned} \tag{7.14}$$

The kinetic terms of the scalars and spinors in hypermultiplets should be related as in

$$\mathcal{L}_{\text{hyper}} = -\tfrac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y - 2\bar{\zeta}^A \mathcal{D}\!\!\!/\zeta_A + \ldots \tag{7.15}$$

This gives a cancellation in leading order (constant metric and vielbein). For local supersymmetry it shows that there is a Noether term

$$\mathcal{L}_{\text{Noether,hyper}} = -\mathrm{i}\bar{\zeta}^A\gamma^\mu(\partial\!\!\!/ q^X)\psi^i_\mu f_{XiA} + \text{h.c.}. \tag{7.16}$$

7.4 Algebra and gravitino transformation

We calculate the algebra of gauge transformations on the hyperscalars. We obtain (using only the left supersymmetry chirality)

$$[\delta_G,\delta_Q]q^X = \mathrm{i}g\nabla_Y f^X_{iA}\bar{\epsilon}^i\zeta^A k^Y_I - \mathrm{i}gf^Y_{iA}\nabla_Y k^X_I\bar{\epsilon}^i\zeta^A - \mathrm{i}f^X_{iA}\bar{\epsilon}^i\delta_G\zeta^A. \tag{7.17}$$

The derivatives are at first ordinary derivatives, but the Levi-Civita connection cancels between the two covariant derivatives. For the gauge transformation of the hyperini, use [49, (2.94)]. Note that this mentions $\widehat{\delta}$. The difference with δ cancels with the $\mathrm{Sp}(n_H)$ connection in the derivative of the vielbein. For $\nabla_Y k^X_I$ one uses [49, (B.81)], and

$$f^Y_{iA}L_Y{}^X{}_B{}^C = f^X_{iB}\delta^C_A, \qquad f^Y_{iA}J^x{}_Y{}^X = -\mathrm{i}(\sigma^x)_i{}^j f^X_{jA} \tag{7.18}$$

The L term then cancels with the transformation of the hyperino. We remain with

$$[\delta_G, \delta_Q]q^X = -\mathrm{i} f^X_{jA} \left[-\nu \mathrm{i} g (\sigma^x)_i{}^j P^x_I - k^Y_I \omega_{Y i}{}^j \right] \bar{\epsilon}^i \zeta^A. \tag{7.19}$$

This gives the algebra of supersymmetry and gauge transformations:

$$[\delta_G(\alpha^I), \delta_Q(\epsilon)] = \delta_Q \left(\epsilon'^i = -\epsilon^j \left(\mathrm{i}\nu g (\sigma^x)_j{}^i P^x_I + k^Y_I \omega_{Y j}{}^i \right) \right). \tag{7.20}$$

This proves the normalization of the term in the gravitino transformation

$$\delta_Q \psi^i_\mu = -\mathrm{i}\nu g (\sigma^x)_j{}^i P^x_I W^I_\mu \epsilon^j. \tag{7.21}$$

7.5 Gauging and potential

We can add fermionic mass terms of the general form

$$e^{-1}\mathcal{L}_{\text{ferm.mass}} = -g S^{ij} \bar{\psi}_{\mu i} \gamma^{\mu\nu} \psi_{\nu j} + \tfrac{1}{2} g g_{\alpha\bar{\beta}} N^\alpha_{ij} \bar{\lambda}^{\bar{\beta} i} \gamma^\mu \psi^j_\mu + 2g \mathcal{N}^{iA} \varepsilon_{ij} \bar{\zeta}_A \gamma^\mu \psi^j_\mu + \text{h.c.}. \tag{7.22}$$

One can then check that variations of the actions with one derivative and linear in fermions cancel if there are transformations of the fermions like

$$\begin{aligned} \delta\psi^i_\mu &= \ldots - g\gamma_\mu S^{ij} \epsilon_j, \\ \delta\lambda^\alpha_i &= \ldots + g N^\alpha_{ij} \epsilon^j, \\ \delta\zeta^A &= \ldots + g \mathcal{N}^{iA} \varepsilon_{ij} \epsilon^j. \end{aligned} \tag{7.23}$$

As the determinant of the vierbein transforms as

$$\delta e = \tfrac{1}{2} e \bar{\epsilon}^i \gamma^\mu \psi_{\mu i} + \text{h.c.}, \tag{7.24}$$

we determine the potential from the variations of the terms in (7.22) as

$$g^{-2} V = -6 S^{ij} S_{ij} + \tfrac{1}{2} g_{\alpha\bar{\beta}} N^\alpha_{ij} N^{\bar{\beta} ij} + 2 \mathcal{N}^{iA} \mathcal{N}_{iA}. \tag{7.25}$$

7.6 Determining the shifts

To calculate the terms in N^α_{ij} related to the moment maps, we consider for now $k^\alpha_I = 0$ and calculate the terms in the variation of the action with λ and one spacetime derivative. The relevant parts of the action are

$$\mathcal{L} = -\tfrac{1}{2} \nabla_\mu q^X g_{XY} \nabla^\mu q^Y - \tfrac{1}{2} g_{\alpha\bar{\beta}} \bar{\lambda}^\alpha_i \not{\nabla} \lambda^{\bar{\beta} i} + x \bar{\zeta}_A k^X_I f^{iA}_X \mathcal{D}_{\bar{\alpha}} \bar{X}^I \lambda^{\bar{\alpha} j} \varepsilon_{ij}, \tag{7.26}$$

where x is to be determined, and we write the terms relevant for ϵ^i variations. The variation of the first term is the variation of the gauge field W_μ in the covariant derivative. This gives

$$\begin{aligned} \delta\mathcal{L} = & -\tfrac{1}{2} \partial_\mu q^X k_{XI} \mathcal{D}_{\bar{\alpha}} \bar{X}^I \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \lambda^{\bar{\alpha} j} + \tfrac{1}{2} g_{\alpha\bar{\beta}} \left(\partial_\mu N^\alpha_{ij} \right) \bar{\epsilon}^j \gamma^\mu \lambda^{\bar{\beta} i} \\ & + \frac{\mathrm{i}x}{4} \left(J_{XYk}{}^i + g_{XY} \delta_k{}^i \right) \bar{\epsilon}^k (\not{\partial} q^Y) \lambda^{\bar{\alpha} j} \varepsilon_{ij} k^X_I \mathcal{D}_{\bar{\alpha}} \bar{X}^I \end{aligned} \tag{7.27}$$

A $\partial_\mu\epsilon$ term has been neglected to get to the second term, but that is provided by the second term of (7.22).

This gives $x = -2\mathrm{i}$ and

$$g_{\alpha\bar\beta}\partial_Y N_{ij}^\alpha = -\mathcal{D}_{\bar\beta}\bar X^I k_I^X J_{XYij} = -2\mathcal{D}_{\bar\beta}\bar X^I \partial_Y P_{Iij}. \tag{7.28}$$

(Note that we have here only leading terms, thus $\partial = \nabla$). We thus obtain

$$N_{ij}^\alpha = -2g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar X^I P_{Iij}. \tag{7.29}$$

Similarly we can calculate the terms with $\zeta\partial q$. The terms that contribute are

$$-2\bar\zeta^A \not{\mathcal{D}}\zeta_A + 2g\mathcal{N}^{iA}\varepsilon_{ij}\bar\zeta_A\gamma^\mu\psi_\mu^j + g\mathcal{M}^{AB}\bar\zeta_A\zeta_B + \mathrm{i}\bar\zeta_A\gamma^\mu(\not\partial q^X)\psi_{\mu i}f_X^{iA}, \tag{7.30}$$

where $\mathcal{M}^{AB}$ still has to be determined. We find the equation

$$-2g\partial_X\mathcal{N}^{iA}\varepsilon_{ij} - \mathrm{i}g\mathcal{M}^{AB}f_{XjB} + 2\mathrm{i}g f_X^{iA}S_{ij}. \tag{7.31}$$

Assume now

$$\mathcal{N}^{iA} = -\mathrm{i}f_X^{iA}k_I^X\bar X^I, \tag{7.32}$$

then using again [49, (B.81)], and similar equations as (7.18), we obtain

$$\mathcal{M}^{AB} = -2t_I^{AB}\bar X^I, \qquad S_{ij} = -P_{Iij}\bar X^I. \tag{7.33}$$

As a further check we now consider the $\epsilon\partial\psi$ terms in the variation of the action. The terms in the action that contribute are

$$\begin{aligned}\mathcal{L} \;=\;& -\bar\psi_\mu^i\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_{\rho i} - gS_{ij}\bar\psi_\mu^i\gamma^{\mu\nu}\psi_\nu^j + 2g\mathcal{N}^{iA}\varepsilon_{ij}\bar\zeta_A\gamma^\mu\psi_\mu^j - \mathrm{i}\bar\zeta^A\gamma^\mu(\not\partial q^X)\psi_\mu^i f_{XiA}\\ & -\tfrac12 g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y.\end{aligned} \tag{7.34}$$

The last term is relevant for the covariantization and transformation of vectors [last term in (7.8)] We obtain for the variations

$$\begin{aligned}\delta\mathcal{L} \;=\;& 2g\bar\psi_\mu^i\gamma^{\mu\nu}\epsilon^j\partial_\nu S_{ij} + \mathrm{i}g\mathcal{N}^{iA}\varepsilon_{ij}f_{XkA}\left[\bar\epsilon^k(\not\partial q^X)\gamma^\mu\psi_\mu^j - \bar\epsilon^j\gamma^\mu(\not\partial q^X)\psi_\mu^k\right]\\ & -(\partial_\mu q^X)k_{IX}\varepsilon_{ij}\bar\epsilon^i\psi_\mu^j\bar X^I.\end{aligned} \tag{7.35}$$

With (7.32) we have

$$2\mathrm{i}g\mathcal{N}^{iA}\varepsilon_{ij}f_{XkA} = \left[2\nabla_X P_{Ikj} + k_{IX}\varepsilon_{kj}\right]\bar X^I. \tag{7.36}$$

The last term arranges the cancellation with the second line in (7.35). This is consistent with (7.33).

Table 5: *Multiplets and fields of the super-Poincaré theories*

spin	pure SG	vector mult.	hypermult.	field	indices
2	1			e_μ^a	$\mu, a = 0, \ldots, 4$
$\frac{3}{2}$	2			ψ_μ^i	$i = 1, 2$
1	1	n		A_μ^I	$I = 0, \ldots, n$
$\frac{1}{2}$		$2n$	$2r$	λ_i^x, ζ^A	$A = 1, \ldots, 2r$
0		n	$4r$	ϕ^x, q^X	$x = 1, \ldots, n$; $X = 1, \ldots, 4r$

8 Quaternionic and very special geometry.

This section will first deal with matter couplings in 5 dimensions. Then we will relate the theories in 4 and 5 dimensions with some maps.

We consider general couplings with n vector multiplets and r scalar multiplets. Table 5 gives their content, the names that we use for the fields, and the corresponding range of indices. In the superconformal method, these are obtained in a different way. One starts with the Weyl multiplet, and adds vector multiplets and hypermultiplets in representations of the superconformal algebra. As well for the vector multiplets as for the hypermultiplets, one starts by adding one more multiplet than appears in the final super-Poincaré theory. These 'compensating multiplets' contain the degrees of freedom that will be gauge-fixed. This is schematically represented in table 6. It is indicated how the superfluous symmetries are fixed,

Table 6: *Multiplets and fields in the superconformal construction*

spin	Weyl	vector	hyper	gauge fix	auxiliary
2	e_μ^a				
$\frac{3}{2}$	ψ_μ^i				
	$V_{\mu i}{}^j$, T_{ab}				auxiliary
1		$n+1$			
$\frac{1}{2}$	χ^i	$2(n+1)$	$2(r+1)$	2: S	χ^i with 2 others
0	D	$n+1$	$4(r+1)$	1: dilatations, 3: $SU(2)$	D and 1 other

and how some of the fields of the Weyl multiplet serve as Lagrange multipliers eliminating degrees of freedom of the spin 1/2 and scalar fields. The field $V_{\mu i}{}^j$ will be eliminated by its field equation, and will play the role of $SU(2)$ curvature of the quaternionic manifold defined by the hyperscalars. The field T_{ab} will become a function of the field strengths of the vectors in the vector multiplet (dressed by the scalars), and plays the role of gauge field that enters in the gravitino transformation (related to the central charge).

8.1 Very special real and quaternionic-Kähler manifolds

The manifolds of supergravity–matter couplings in $d = 5$ are similar to those that are known from $N = 2$ in 4 dimensions. Table 5 would be nearly identical for 4 dimensions, except that each vector multiplet then contains two scalars. The supersymmetry defines a complex structure, and the manifold is Kählerian. In $N = 1$ supergravity, general Kähler manifolds are possible. In $N = 2$ they are restricted to a category that is called 'special Kähler manifolds' [1]. The quartets of scalars in hypermultiplets are connected by 3 complex structures and the manifold is quaternionic-Kähler [107]. Another recent review containing the fundamental facts of these manifolds is given in [108].

8.1.1 Very special real manifolds

We first consider the vector multiplets [88]. In 5 dimensions, these have real scalars (one of the scalars of 4 dimensions sits in the 5d-vector). We define 'very special real manifolds' [79] as those that appear in these couplings of vector multiplets to 5-dimensional supergravity. It is clear from the above, that they can be described in superconformal tensor calculus by starting with $n+1$ scalars, which we denote h^I, as in table 6. Then we impose a dilatational gauge choice. This defines an n-dimensional hypersurface in the $(n+1)$-dimensional space.

The locally supersymmetric action of the vector multiplets in 5 dimensions contains always a Chern–Simons term of the form $\mathcal{C}_{IJK}A^I\mathrm{d}A^J\mathrm{d}A^K$. In order for this to be gauge-invariant, the $\mathcal{C}_{IJK}$ have to be constant. This tensor is completely symmetric in its indices, and supersymmetry implies that the full action is determined by these constants (up to the choice of coordinates on the manifold). Thus the set of numbers $\mathcal{C}_{IJK}$ are all one needs to specify a very special real manifold [88]. For an arbitrary set, one still has to verify whether they allow a non-empty domain with positive-definite metric on the scalar manifold.

The dilatational gauge choice that is most appropriate is the condition

$$\mathcal{C}_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1\,. \tag{8.1}$$

ϕ^x are coordinates on this manifold such that the embedding $h(\phi)$ satisfies the condition. The metric on the scalar manifold is then

$$g_{xy} = -3(\partial_x h^I)(\partial_y h^J)\mathcal{C}_{IJK}h^K\,. \tag{8.2}$$

8.1.2 Quaternionic-Kähler manifolds

Let us now look at the other side. The hypermultiplets were introduced in section 4.2.3. The scalar fields define a $4r$-dimensional manifold with coordinates q^X, and the tangent space contains vectors are labelled with indices (iA). We showed how the supersymmetry algebra leads to a hypercomplex structure. In order that we can construct an action, we need an invariant metric, and this promotes the manifold to a hyper-Kähler manifold. The requirement of conformal symmetry further restricts the manifold. The gauge fixing then leads to a projection on a submanifold, which turns out to be quaternionic-Kähler.

Quaternionic-Kähler manifolds entered physics in [107], and [109] contains a lot of interesting properties. There were two workshops on quaternionic geometry where mathematics and physics results were brought together [110, 111]. Other recent papers that review the properties of quaternionic manifolds are [108, 112, 49].

Quaternionic manifolds Quaternionic manifolds have a torsionless connection $\Gamma^Z_{XY} = \Gamma^Z_{YX}$. On the tangent space there is a connection

$$\Omega_{X\,jB}{}^{iA} \equiv f^Y_{jB}\left(\partial_X f^{iA}_Y - \Gamma^Z_{XY} f^{iA}_Z\right) = -\omega_{Xj}{}^i \delta_B{}^A - \omega_{XB}{}^A \delta_j{}^i\,, \tag{8.3}$$

where $\omega_{Xj}{}^i$ is traceless. If this $\Omega_{X\,jB}{}^{iA}$, for each X, would be a general $4r \times 4r$ matrix, then we would say that the holonomy is not restricted (or sits in $\mathrm{G}\ell(4r)$). The splitting as in the right-hand side of this equation implies that the holonomy group is restricted to $\mathrm{SU}(2) \times \mathrm{G}\ell(r,\mathbb{H})$. We can use (8.3) to write a statement of covariant constancy of the vielbein:

$$\partial_X f^{iA}_Y - \Gamma^Z_{XY} f^{iA}_Z + f^{jA}_Y \omega_{Xj}{}^i + f^{iB}_Y \omega_{XB}{}^A = 0\,, \tag{8.4}$$

with composite gauge fields for $\mathrm{SU}(2)$ and $\mathrm{G}\ell(r,\mathbb{H})$. These conditions promote the almost quaternionic structure to a quaternionic structure, and the manifolds is 'quaternionic'. If the $\mathrm{SU}(2)$ connection is zero, they are called 'hypercomplex', which is the case before the gauge fixing (or in rigid supersymmetry).

The integrability condition of (8.4), (multiplied by a vielbein) is

$$R^Z{}_{WXY} = f^Z_{iA} f^{jA}_W \mathcal{R}_{XYj}{}^i + f^Z_{iA} f^{iB}_W \mathcal{R}_{XYB}{}^A = -J_W{}^{Z\alpha} \mathcal{R}_{XY}{}^\alpha + f^Z_{iA} f^{iB}_W \mathcal{R}_{XYB}{}^A\,, \tag{8.5}$$

where respectively the metric curvature $R^Z{}_{WXY} \equiv 2\partial_{[X}\Gamma^Z_{Y]W} + 2\Gamma^Z_{V[X}\Gamma^V_{Y]W}$, the $SU(2)$ curvature $\mathcal{R}_{XY}{}^\alpha \equiv 2\partial_{[X}\omega_{Y]}{}^\alpha + 2\omega_{[X}{}^\beta \omega_{Y]}{}^\gamma \varepsilon^{\alpha\beta\gamma}$, and the $\mathrm{G}\ell(r,Q)$ curvature $\mathcal{R}_{WXB}{}^A$ appear.

Quaternionic-Kähler manifolds. Quaternionic-Kähler manifolds (which include 'hyper-Kähler manifolds' in the case that the $\mathrm{SU}(2)$ curvature vanishes) by definition have a metric.

For $r > 1$ one can prove that these manifolds are Einstein, and that the $\mathrm{SU}(2)$ curvatures are proportional to the complex structures:

$$R_{XY} = \frac{1}{4r} g_{XY} R\,, \qquad \mathcal{R}^\alpha_{XY} = \tfrac{1}{2}\nu J^\alpha_{XY}\,, \qquad \nu = \frac{1}{4r(r+2)} R\,. \tag{8.6}$$

(with $R_{XY} = R^Z{}_{XZY}$). For $r = 1$ this is part of the definition of quaternionic-Kähler manifolds. Hyper-Kähler manifolds are those where the $\mathrm{SU}(2)$ curvature is zero, and these are thus also Ricci-flat.

Supergravity In supergravity we find all these constraints from requiring a supersymmetric action. Moreover, we need for the invariance of the action that the last equation of (8.6) is satisfied with $\nu = -1$. This implies that the scalar curvature is $R = -4r(r+2)$. This excludes e.g. the compact symmetric spaces.

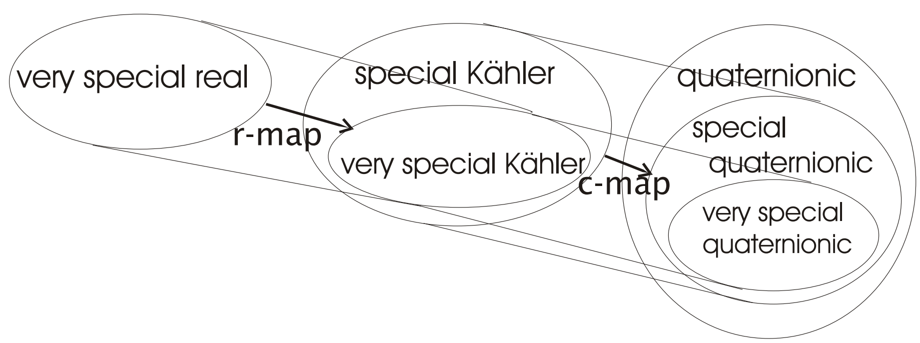
8.1.3 The family of special manifolds

We now place these manifolds in the context of the manifolds that are obtained for supersymmetries with 8 real supercharges. Note that a higher number of supercharges would restrict the possibilities for the scalar manifolds to a discrete number of symmetric spaces.

We first consider vector multiplets in 5 or 4 dimensions with $N = 2$. Vector multiplets in 6 dimensions do not contain scalars. When reducing to 3 dimensions, the vectors become dual to scalars (we can perform duality transformations as we are just considering kinetic terms here, and we can thus restrict to Abelian vectors). Therefore the multiplet in 3 dimensions is dual to a multiplet with only scalars: the hypermultiplet. For hypermultiplets, the spacetime dimension is not really relevant as there are no vectors, and thus the results for hypermultiplets are the same for any dimension. In the picture it is convenient to consider them in 3 dimensions because of the dimensional reduction that we just described. With real scalars in the vector multiplets in 5 dimensions, these geometries are real geometries, those in 4 dimensions are Kählerian, and the hypermultiplets lead to 3 complex structures. Furthermore, we can distinguish between theories that appear in rigid supersymmetry, and those in supergravity. This leads to the overview in the upper part of table 7. The geometries

Table 7: *Geometries from supersymmetric theories with 8 real supercharges, and the connections provided by the* **r***-map and the* **c***-map.*

	$D=5$ vector multiplets	$D=4$ vector multiplets	hypermultiplets
rigid (affine)	affine very special real	affine special Kähler	hyperkähler
local (projective)	(projective) very special real	(projective) special Kähler	quaternionic-Kähler



that are related to rigid supersymmetry have been called 'affine' in the mathematics literature [89, 113], while those for supergravity are called 'projective' (and these are the default, in the sense that e.g. special Kähler refers to the geometry that is found in supergravity). The analogous manifolds with 3 complex structures got already a name in the literature: the ones that occur in rigid supersymmetry are the hyperkähler manifolds, while those in supergravity are the quaternionic-Kähler manifolds[27]

[27]Mathematicians include hyperkähler as a special case of what they call 'quaternionic-Kähler', while physicists reserve the name quaternionic to the manifolds that have non-vanishing SU(2) curvature, which excludes the hyperkähler ones. Furthermore, we will restrict ourselves to the quaternionic-Kähler manifolds of negative scalar curvature, as those are the only ones that appear in supergravity. For manifolds with continuous isometries, this implies that they are non-compact.

The name 'projective' versus 'affine' can be understood from the construction of the manifolds in supergravity using superconformal tensor calculus. We saw already (see table 6) how the real very special manifolds are obtained starting from $(n+1)$ vector multiplets. Before any gauge fixing, these are just real manifolds with a dilatational invariance. This manifold has therefore a cone structure, with $\mathcal{C}_{IJK}h^I h^J h^K$ as the radial coordinate. The physical scalars of the supergravity theory are thus defined modulo this dilatational scaling, and the manifolds that occur in supergravity can be seen as a projective space of dimension n.

Similarly, to construct special Kähler geometry, one starts in 4 dimensions with the couplings as they occur in rigid supersymmetry, demanding the presence of a superconformal symmetry. Again, the manifold has a cone structure, and the dilatational gauge condition selects a submanifold at fixed radius. In this case, the superconformal group contains a U(1) invariance and the manifold at fixed radius is a 'Sasakian manifold' of dimension $2n+1$, if this U(1) is not gauged. In conformal supergravity the U(1) is local and eliminates one more scalar. The gauge field of this U(1), which is an auxiliary field in the superconformal tensor calculus (similar to $V_{\mu i}{}^j$ in table 6), becomes by its field equation the $U(1)$ connection on the Kähler manifold. The final manifold in super-Poincaré has then non-trivial $U(1)$ curvature (and will be a Hodge-Kähler manifold).

The construction for quaternionic manifolds is similar, as has been demonstrated recently in 4 dimensions in [114]. One starts then from hyperkähler cones. The dilatational gauge choice leads to a tri-Sasakian manifold of dimensions $4r+3$ for ungauged $SU(2)$. The $SU(2)$ gauge fields of the Weyl multiplet get by their field equations the value $V_{\mu i}{}^j = \partial_\mu q^X \omega_{Xi}{}^j$, using the $SU(2)$ connection that we had in the previous section. The $SU(2)$ curvature is thus non-zero as required by (8.6).

Dimensional reduction gives a mapping between these manifolds. These mappings have been called the **c**-map (from special Kähler to special quaternionic) [78], and the **r**-map (from very special real to very special Kähler) [115]. They are represented in the lower part of table 7. Dimensional reduction of a manifold in 5 dimensions gives a 4-dimensional theory. But the 4-dimensional theories that can be obtained in this way, are only a subset of all 4-dimensional theories. The table shows the structure in the names given to various classes of manifolds. Very special Kähler manifolds are a subset of all special Kähler manifolds. The quaternionic manifolds that are in the image of the **c**-map are the special quaternionic manifolds, and those in the image of the **c**∘**r**-map are the very special quaternionic manifolds. It is remarkable that nearly all the homogeneous quaternionic manifolds are very special quaternionic manifolds [115] (The only non-special homogeneous quaternionic manifolds are the quaternionic projective spaces).

8.2 Old on c-map

The **c** map [78] gives a mapping from a special Kähler to a quaternionic manifold. It is induced by reducing an $N=2$ supergravity action in $d=4$ spacetime dimensions to an action in $d=3$ spacetime dimensions, by suppressing the dependence on one of the (spatial) coordinates. The resulting $d=3$ supergravity theory can be written in terms of $d=3$ fields and this rearranges the original fields such that the number of scalar fields increases from

$2n$ to $4(n+1)$. This map is also obtained in string theory context by changing from a type IIA to a type IIB string or vice-versa.

This leads to the notion of '*special quaternionic manifolds*', which are those manifolds appearing in the image of the **c** map. They are a subclass of the quaternionic manifolds. Similarly, very special real manifolds are the manifolds defined by coupling (real) scalars to vector multiplets in 5 dimensions [88] (characterized by a symmetric tensor d_{ABC}). Very special Kähler manifolds [79] are induced as the image under the **r** map (dimensional reduction from 5 to 4 dimensions) and very special quaternionic manifolds as the image of the $\mathbf{c}\circ\mathbf{r}$ map.

How the vector multiplets in $d=5$ with $n-1$ real scalars are related to $d=4$ multiplets with n complex scalars is schematically shown in table 8, and the relation with $d=3$

Table 8: *The* **r** *map induced by dimensional reduction from* $d=5$ *to* $d=4$ *supergravity. The number of fields of integer spins is indicated.*

$d=5$ spins	2	1	0
numbers	1	n	$n-1$
$d=4$ spins			
2	1		
1	1	n	
0	1	n	$n-1$

hypermultiplets with $n+1$ quaternions is shown in table 9.

Table 9: *The* **c** *map as dimensional reduction from* $d=4$ *to* $d=3$ *supergravity. The number of fields of various spins is indicated and names are assigned to the scalar fields in* $d=3$.
.

$d=4$ spins	2	1	0
numbers	1	$n+1$	$2n$
$d=3$ spins			
2	1		
0	2	$2(n+1)$	$2n$
	ϕ,σ	A^I, B_I	$z^A, \bar{z}^A$

8.3 Homogeneous and symmetric spaces.

Homogeneous and symmetric spaces are the most known manifolds. These are spaces of the form G/H, where G are the isometries and H is its isotropy subgroup. The group G is not necessarily a semi-simple group, and thus not all the homogeneous spaces have a clear name. The symmetric spaces are those for which the algebra splits as $g = h + k$ and all commutators $[k, k] \subset h$. The homogeneous special manifolds are classified in [115].

It turns out that homogeneous special manifolds are in one-to-one correspondence to realizations of real Clifford algebras with signature $(q + 1, 1)$ for real, $(q + 2, 2)$ for Kähler, and $(q + 3, 3)$ for quaternionic manifolds. Thus, the spaces are identified by giving the number q, which specifies the Clifford algebra, and by specifying its representation. If q is not a multiple of 4, then there is only one irreducible representations, and we thus just have to mention the multiplicity P of this representation. The spaces are denoted as $L(q, P)$. If $q = 4m$ then there are two inequivalent representations, chiral and antichiral, and the spaces are denoted as $L(q, P, \dot{P})$. If we use n as the complex dimension of the special Kähler space,

Table 10: *Homogeneous manifolds. In this table, q, P, $\dot{P}$ and m denote positive integers or zero, and $q \neq 4m$. SG denotes an empty space, which corresponds to supergravity models without scalars. Furthermore, $L(4m, P, \dot{P}) = L(4m, \dot{P}, P)$, while $L(0, n) = L(n, 0)$. The horizontal lines separate spaces of different rank. The first non-empty space in each column has rank 1. Going to the right or down a line increases the rank by 1. The manifolds indicated by a $\star$ did not get a name. The number n is the complex dimension of the Kähler space, and is related to the other dimensions as in (8.7), where the n is also given for the general cases in the last two rows.*

	n	real	Kähler	quaternionic
$L(-3, P)$	P			$\frac{\mathrm{USp}(2P+2,2)}{\mathrm{USp}(2P+2)\otimes\mathrm{SU}(2)}$
SG_4	0		SG	$\frac{\mathrm{U}(1,2)}{\mathrm{U}(1)\otimes\mathrm{U}(2)}$
$L(-2, P)$	$1 + P$		$\frac{\mathrm{U}(P+1,1)}{\mathrm{U}(P+1)\otimes\mathrm{U}(1)}$	$\frac{\mathrm{SU}(P+2,2)}{\mathrm{SU}(P+2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)}$
SG_5	1	SG	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{G_2}{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}$
$L(-1, P)$	$2 + P$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$	$\star$	$\star$
$L(4m, P, \dot{P})$		$\star$	$\star$	$\star$
$L(q, P)$		$X(P, q)$	$H(P, q)$	$V(P, q)$

the dimension of these manifolds is ($\dot{P} = 0$ if $q \neq 4m$)

$$n = 3 + q + (P + \dot{P})\mathcal{D}_{q+1}\,, \qquad \left\{ \begin{array}{l} \dim_{\mathbb{R}}[\text{very special real } L(q, P, \dot{P})] = n - 1 \\ \dim_{\mathbb{R}}[\text{special Kähler } L(q, P, \dot{P})] = 2n \\ \dim_{\mathbb{R}}[\text{quaternionic-Kähler } L(q, P, \dot{P})] = 4(n + 1). \end{array} \right. \tag{8.7}$$

where $\mathcal{D}_{q+1}$ is the dimension of the irreducible representation of the Clifford algebra in $q + 1$

dimensions with positive signature, i.e.

$$\mathcal{D}_{q+1} = 1 \ \text{ for } q = -1, 0\,, \qquad \mathcal{D}_{q+1} = 2 \ \text{ for } q = 1\,, \qquad \mathcal{D}_{q+1} = 4 \ \text{ for } q = 2\,,$$
$$\mathcal{D}_{q+1} = 8 \text{ for } q = 3, 4\,, \qquad \mathcal{D}_{q+1} = 16 \text{ for } q = 5, 6, 7, 8\,, \qquad \mathcal{D}_{q+8} = 16\,\mathcal{D}_q\,. \qquad (8.8)$$

The very special manifolds are defined by coefficients $\mathcal{C}_{IJK}$ as we will see below. For the homogeneous ones, we can write them as

$$\mathcal{C}_{IJK}\, h^I h^J h^K = 3\left\{ h^1 \left(h^2\right)^2 - h^1 \left(h^\mu\right)^2 - h^2 \left(h^i\right)^2 + \gamma_{\mu ij}\, h^\mu\, h^i\, h^j \right\}. \qquad (8.9)$$

We decomposed the indices $I = 1, \ldots, n$ into $I = 1, 2, \mu, i$, with $\mu = 1, \ldots, q+1$ and $i = 1, \ldots, (P + \dot{P})\mathcal{D}_{q+1}$. Here, $\gamma_{\mu ij}$ is the $(q+1, 0)$ Clifford algebra representation that we mentioned. Note that these models have predecessors in 6 dimensions, with $q+1$ tensor multiplets and $(P + \dot{P})\mathcal{D}_{q+1}$ vector multiplets. The gamma matrices are then the corresponding coupling constants between the vector and tensor multiplets.

Considering further the table 10, we find in the quaternionic spaces the homogeneous ones that were found in [116], together with those that were discovered in [115] (the ones with a $\star$ except for the series $L(0, P, \dot{P})$, which were already in [116], and denoted there as $W(P, \dot{P})$).

Observe that the classification of homogeneous spaces exhibits that the quaternionic projective spaces have no predecessor in special geometry, and that the complex projective spaces have no predecessor in very special real manifolds. Similarly, only those with $q \geq -1$ can be obtained from 6 dimensions [with the scalars of tensor multiplets describing $\mathrm{SO}(1, q+1)/\mathrm{SO}(q+1)$] and $L(-1, 0)$ corresponds to pure supergravity in 6 dimensions.

There are still symmetric spaces in the range $q \geq -1$. These are shown in table 11. For

Table 11: *Symmetric very special manifolds. Note that the very special real manifolds $L(-1, P)$ are symmetric, but not their images under the* **r** *map. The number n is the dimension as in table 10.*

	n	real	Kähler	quaternionic
$L(-1,0)$	2	$\mathrm{SO}(1,1)$	$\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^2$	$\frac{\mathrm{SO}(3,4)}{(\mathrm{SU}(2))^3}$
$L(-1,P)$	$2+P$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$		
$L(0,P)$	$3+P$	$\mathrm{SO}(1,1)\otimes\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\otimes\frac{\mathrm{SO}(P+2,2)}{\mathrm{SO}(P+2)\otimes\mathrm{SO}(2)}$	$\frac{\mathrm{SO}(P+4,4)}{\mathrm{SO}(P+4)\otimes\mathrm{SO}(4)}$
$L(1,1)$	6	$\frac{\mathrm{S}\ell(3,\mathbb{R})}{\mathrm{SO}(3)}$	$\frac{\mathrm{Sp}(6)}{\mathrm{U}(3)}$	$\frac{F_4}{\mathrm{USp}(6)\otimes\mathrm{SU}(2)}$
$L(2,1)$	9	$\frac{\mathrm{S}\ell(3,C)}{\mathrm{SU}(3)}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\otimes\mathrm{SU}(3)\otimes\mathrm{U}(1)}$	$\frac{E_6}{\mathrm{SU}(6)\otimes\mathrm{SU}(2)}$
$L(4,1)$	15	$\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\otimes\mathrm{U}(1)}$	$\frac{E_7}{\mathrm{SO}(12)\otimes\mathrm{SU}(2)}$
$L(8,1)$	27	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes\mathrm{U}(1)}$	$\frac{E_8}{E_7\otimes\mathrm{SU}(2)}$

the symmetric special Kähler spaces, this reproduces the classification obtained in [70]. A study of the full set of isometries could be done systematically in these models. All this has been summarised in [117].

9 Final results

In this section, we will summarize the final action and transformation laws. We will restrict here to the action for the bosons and transformation laws of the fermions, as these are the main tools for applications. This section repeats all definitions as it is meant to be readable by itself.

9.1 6 dimensions

9.2 5 dimensions

We consider the theory with

- Supergravity including the vielbein e_μ^a and gravitino ψ_μ^i. The graviphoton is included in the vector multiplets.

- Vector-tensor multiplets enumerated by $\tilde{I} = 0, \ldots, n_V + n_T$ where n_V is the number of vector multiplets and n_T is the number of tensor multiplets. The index is further split as $\tilde{I} = (I, M)$, where $I = 0, \ldots, n_V$ and $M = n_V + 1, \ldots, n_V + n_T$. The vector or tensor fields are grouped in

$$H_{\mu\nu}^{\tilde{I}} \equiv \left(F_{\mu\nu}^I, \widetilde{B}_{\mu\nu}^M \right), \qquad F_{\mu\nu}^I \equiv 2\partial_{[\mu} A_{\nu]}^I + g f_{JK}{}^I A_\mu^J A_\nu^K. \tag{9.1}$$

  where $\widetilde{B}_{\mu\nu}^M$ are the fundamental tensor fields and A_μ^I are the fundamental vector fields, gauging an algebra with structure constants $f_{JK}{}^I$ and gauge coupling constant g. The fermions of these multiplets are denoted as λ^{xi} and the real scalars as ϕ^x where $x = 1, \ldots, n_V + n_T$. The couplings of the vector and tensor multiplets are determined by the constants $\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}$, an antisymmetric and invertible metric Ω_{MN} and the transformation matrices $t_{I\tilde{J}}{}^{\tilde{K}}$ related by (we kept here κ as the gravitational coupling constant, which has been put equal to 1 in other places of this text)

$$\begin{aligned} &\mathcal{C}_{M\tilde{J}\tilde{K}} = \sqrt{\frac{3}{8\kappa^2}} t_{(\tilde{J}\tilde{K})}{}^P \Omega_{PM}, \qquad t_{I[M}{}^P \Omega_{N]P} = 0, \qquad t_{I(\tilde{J}}{}^{\widetilde{M}} \mathcal{C}_{\widetilde{K}\widetilde{L})\widetilde{M}} = 0, \\ &(t_M)_{\tilde{J}}{}^{\tilde{K}} = 0, \qquad (t_I)_{\tilde{J}}{}^{\tilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix}. \end{aligned} \tag{9.2}$$

  Note that this implies that at least one index of a non-zero $\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}$ should correspond to a vector multiplet. The above-mentioned scalars are a parametrization of the manifold defined by

$$\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}}(\phi) h^{\tilde{J}}(\phi) h^{\tilde{K}}(\phi) = 1. \tag{9.3}$$

- n_H hypermultiplets with scalars q^X and spinors ζ^A, where $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$. Their interactions are determined by the vielbeins, f_X^{iA}, invertible as $4n_H \times 4n_H$ matrices.

The bosonic action is

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm bos} &= \tfrac{1}{2\kappa^2}R - \tfrac14 a_{\tilde I\tilde J}H^{\tilde I}_{\mu\nu}H^{\tilde J\mu\nu} - \tfrac12 g_{xy}\mathcal{D}_\mu\phi^x\mathcal{D}^\mu\phi^y - \tfrac12 g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y - V \\ &\quad + \frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde B^M_{\mu\nu}\Big(\partial_\rho\widetilde B^N_{\sigma\tau} + 2gt_{IJ}{}^N A^I_\rho F^J_{\sigma\tau} + gt_{IP}{}^N A^I_\rho\widetilde B^P_{\sigma\tau}\Big) \\ &\quad + \tfrac{\kappa}{12}\sqrt{\tfrac23}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\mathcal{C}_{IJK}A^I_\mu\left[F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac12 gF^K_{\rho\sigma} + \tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right] \\ &\quad -\tfrac18 e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac12 gF^K_{\rho\sigma} + \tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right). \end{aligned} \tag{9.4}$$

The metrics for the vectors and the vector-scalars are defined by

$$\begin{aligned} a_{\tilde I\tilde J} &= -2\mathcal{C}_{\tilde I\tilde J\tilde K}h^{\tilde K} + 3h_{\tilde I}h_{\tilde J}, \qquad h_{\tilde I} \equiv \mathcal{C}_{\tilde I\tilde J\tilde K}h^{\tilde J}h^{\tilde K} = a_{\tilde I\tilde J}h^{\tilde J} \\ g_{xy} &= h^{\tilde I}_x h^{\tilde J}_y a_{\tilde I\tilde J}, \qquad h^{\tilde I}_x \equiv -\sqrt{\frac{3}{2\kappa^2}}\partial_x h^{\tilde I}(\phi). \end{aligned} \tag{9.5}$$

Many useful relations are given in appendix D (where the gravitational coupling constant κ has been put equal to 1.). The domain of the variables should be limited to $h^I(\phi)\neq 0$ and the metrics a_{IJ} and g_{xy} should be positive definite. Due to relation (D.7) the latter two conditions are equivalent.

The metric for the hyperscalars is

$$g_{XY} = f^{iA}_X f^{jB}_Y \varepsilon_{ij}C_{AB}. \tag{9.6}$$

Here C_{AB} is an antisymmetric constant (a generalization to covariantly constant is possible) invertible metric, which governs also the reality properties of the vielbeins:

$$\left(f^{iA}_X\right)^* = f^{jB}_X\varepsilon_{ji}\rho_{BA}, \qquad \rho_{BA}\equiv C_{BC}d^C{}_A, \qquad \rho_{AB}(\rho_{BC})^* = -\delta_A{}^C. \tag{9.7}$$

Here a positive definite (for positive kinetic energies) hermitian matrix $d^A{}_B$ enters and the last reality property is necessary for consistency. The normalization should be such that the scalar curvature of the metric is

$$R = -4n_H(n_H+2)\kappa^2. \tag{9.8}$$

All other quantities in (9.4) are related to gauged symmetries. The gauge symmetry transformations (with parameters α^I) of the bosons are

$$\begin{aligned} \delta_G(\alpha)A^I_\mu &= \partial_\mu\alpha^I - g\alpha^J f_{JK}{}^I A^K_\mu, \qquad \delta_G(\alpha)\widetilde B^M_{\mu\nu} = -g\alpha^J t_{J\tilde K}{}^M H^{\tilde I}_{\mu\nu}, \\ \delta_G h^{\tilde I}(\phi) &= -g\alpha^J t_{J\tilde K}{}^{\tilde I}h^{\tilde K} \\ \delta_G\phi^x &= -g\alpha^I K^x_I, \qquad K^x_I \equiv -\frac1\kappa\sqrt{\frac32}t_{I\tilde J}{}^{\tilde K}h^{\tilde Jx}h_{\tilde K}, \qquad h^{\tilde Jx}\equiv g^{xy}h^{\tilde J}_y, \\ \delta_G q^X &= -g\alpha^I k^X_I, \end{aligned} \tag{9.9}$$

where k^X_I should be isometries of the quaternionic-Kähler metric g_{XY}, whose commutators are determined by the structure constants $f_{IJ}{}^K$:

$$2k^Y_{[I}\partial_Y k^X_{J]} + f_{IJ}{}^K k^X_K = 0. \tag{9.10}$$

These transformations determine immediately the covariant derivatives in this bosonic truncation:

$$\mathcal{D}_\mu \phi^x = \partial_\mu \phi^x + g A_\mu^I K_I^x, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + g A_\mu^I k_I^X. \tag{9.11}$$

These transformations also determine the scalar potential V:

$$V = \tfrac{g^2}{\kappa^4}\left(4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2W_x W^x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\right). \tag{9.12}$$

The quantities in this expression appear in the transformation laws of the fermions. They are

$$\begin{aligned}
&\vec{P} \equiv \kappa^2 h^I \vec{P}_I, \qquad \vec{P}_x \equiv \kappa^2 h_x^I \vec{P}_I, \qquad \mathcal{N}^{iA} \equiv \frac{\sqrt{6}}{4}\kappa h^I k_I^X f_X^{iA},\\
&W^x \equiv \frac{\sqrt{6}}{4}\kappa h^I K_I^x = -\frac{3}{4} t_{J\tilde{I}}{}^{\tilde{P}} h^J h^{\tilde{I}} h^x_{\tilde{P}} = -\frac{3}{4} t_{J\tilde{I}}{}^{\tilde{P}} h^J h^{\tilde{I}x} h_{\tilde{P}},
\end{aligned} \tag{9.13}$$

The remaining undefined quantity is the triplet moment map $\vec{P}_I(q)$ on the quaternionic-Kähler manifold related to any gauge symmetry. This needs a few elements of quaternionic-Kähler geometry. With the fundamental vielbein f_X^{iA} and its inverse f_{iA}^X one constructs the complex structures

$$\vec{J}_X{}^Y \equiv -\mathrm{i} f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y. \tag{9.14}$$

Furthermore, one defines SU(2) connections on the manifold $\vec{\omega}_X$, by requiring the covariant constancy of the complex structures:

$$0 = \mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma_{XY}{}^W \vec{J}_W{}^Z + \Gamma_{XW}{}^Z \vec{J}_Y{}^W + 2\,\vec{\omega}_X \times \vec{J}_Y{}^Z = 0, \tag{9.15}$$

where the Levi-Civita connection of the metric g_{XY} is used. The curvature of this SU(2) connection is related to the complex structure by

$$\vec{R}_{XY} \equiv 2\partial_{[X}\vec{\omega}_{Y]} + 2\vec{\omega}_X \times \vec{\omega}_Y = -\tfrac{1}{2}\kappa^2 \vec{J}_{XY}. \tag{9.16}$$

This allows us to define the moment maps $\vec{P}_I(q)$ as solutions of

$$-\tfrac{1}{2}\vec{J}_{XY} k_I^Y = \mathfrak{D}_X \vec{P}_I \equiv \partial_X \vec{P}_I + 2\vec{\omega}_X \times \vec{P}_I. \tag{9.17}$$

The algebra of the symmetries implies the 'equivariance condition'

$$2\kappa^2 \vec{P}_I \times \vec{P}_J + \tfrac{1}{2}\vec{J}_{XY} k_I^X k_J^Y - f_{IJ}{}^K \vec{P}_K = 0\,. \tag{9.18}$$

The solutions to these equations are unique if the quaternionic-Kähler manifold is non-trivial:

$$4 n_H \kappa^2 \vec{P}_I = \vec{J}_X{}^Y \mathfrak{D}_Y k_I^X. \tag{9.19}$$

However, for $n_H = 0$ there are still two possible solutions for the moment maps, which are called Fayet–Iliopoulos (FI) terms. First, in the case where the gauge group contains an SU(2) factor, we can have

$$\vec{P}_I = \vec{e}_I \xi, \tag{9.20}$$

where ξ is an arbitrary constant, and $\vec{e}_I$ are constants that are nonzero only for I in the range of the SU(2) factor and satisfy

$$\vec{e}_I \times \vec{e}_J = f_{IJ}{}^K \vec{e}_K, \tag{9.21}$$

in order that (9.18) is verified.

The second case is U(1) FI terms. In this case

$$\vec{P}_I = \vec{e}\,\xi_I, \tag{9.22}$$

where $\vec{e}$ is an arbitrary vector in SU(2) space and ξ_I are constants for the I corresponding to U(1) factors in the gauge group.

The $N=2$ supersymmetry rules of the fermionic fields, up to bilinears in the fermions, are given by

$$\begin{aligned}
\delta\psi^i_\mu &= D_\mu(\omega)\epsilon^i + \tfrac{\mathrm{i}\kappa}{4\sqrt{6}} h_{\tilde{I}} H^{\tilde{I}\nu\rho}\left(\gamma_{\mu\nu\rho} - 4g_{\mu\nu}\gamma_\rho\right)\epsilon^i + \tfrac{1}{\kappa\sqrt{6}}\mathrm{i}gP_j{}^i\gamma_\mu\epsilon^j,\\
\delta\lambda^{xi} &= -\tfrac{\mathrm{i}}{2}\not{\mathcal{D}}\phi^x\epsilon^i + \tfrac{1}{4}\gamma\cdot H^{\tilde{I}} h^x_{\tilde{I}}\epsilon^i + \tfrac{1}{\kappa^2}gP^x{}_j{}^i\epsilon^j + \tfrac{1}{\kappa^2}gW^x\epsilon^i,\\
\delta\zeta^A &= \tfrac{1}{2}\mathrm{i}\gamma^\mu\mathcal{D}_\mu q^X f^{iA}_X\epsilon_i - \tfrac{1}{\kappa}g\mathcal{N}^{iA}\epsilon_i.
\end{aligned} \tag{9.23}$$

Here indices are lowered with the symplectic metrics ε_{ij} and C_{AB} using the NW-SE convention as in (A.39). The new quantities are

$$\begin{aligned}
D_\mu(\omega)\epsilon^i &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon^i + \partial_\mu q^X\omega_{Xj}{}^i\epsilon^j + g\kappa^2 A^I_\mu P_{Ij}{}^i\epsilon^j,\\
P_i{}^j &= \mathrm{i}\vec{P}\cdot\vec{\sigma}_i{}^j,
\end{aligned} \tag{9.24}$$

where the latter translation between doublet and triplet notation applies to all SU(2) vector quantities.

To finish this section, let us discuss some aspects of the R-symmetry, which should clarify how the gauge transformations act on the gravitini and how this is consistent with the supersymmetry algebra. We can write (9.24) as

$$\delta_Q\psi_{\mu i} = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon_i - \mathrm{i}\vec{\mathcal{V}}_\mu\cdot\vec{\sigma}_i{}^j\epsilon_j + \ldots, \tag{9.25}$$

where

$$\vec{\mathcal{V}}_\mu = \partial_\mu q^X\vec{\omega}_X - g\kappa^2 A^I_\mu\vec{P}_I \tag{9.26}$$

corresponds to the value of the auxiliary field of the Weyl multiplet that is the gauge field of the SU(2) in the superconformal algebra.

$$\delta_{\mathrm{SU}(2)}\vec{\mathcal{V}}_\mu = \partial_\mu\vec{\Lambda}_{\mathrm{SU}(2)} + 2\vec{\mathcal{V}}_\mu\times\vec{\Lambda}_{\mathrm{SU}(2)}\,. \tag{9.27}$$

Due to the gauge fixing of this SU(2), this invariance is in the final theory not an independent symmetry, but is a linear combination of all the gauge symmetries, i.e.

$$\vec{\Lambda}_{\mathrm{SU}(2)} = g\left(-\vec{\omega}_X k^X_I + \kappa^2\vec{P}_I\right)\alpha^I\,. \tag{9.28}$$

The gauge symmetries are those that act on the independent fields (hyper-scalars q^X and gauge vectors A_μ^I) as in (9.9). To prove (9.27) from these elementary transformations [118] you need the equations (9.16), (9.17) and (9.18).

The gravitino also transforms under the R-symmetry:

$$\delta_{\mathrm{SU}(2)}\psi_{\mu i} = \mathrm{i}\vec{\Lambda}_{\mathrm{SU}(2)} \cdot \vec{\sigma}_i{}^j \epsilon_j \,. \tag{9.29}$$

This means thus that the gravitino transforms under the gauge symmetries as this expression using (9.28).

It can then be checked that this is consistent with the commutation relation (which is thus implicitly a commutator between gauge transformations and supersymmetry)

$$\left[\delta_Q(\epsilon), \delta_{\mathrm{SU}(2)}(\vec{\Lambda}_{\mathrm{SU}(2)})\right] = \delta_Q(\mathrm{i}\vec{\Lambda}_{\mathrm{SU}(2)} \cdot \vec{\sigma}_i{}^j \epsilon_j) \,. \tag{9.30}$$

9.3 4 dimensions

We consider the theory with

- Supergravity including the vielbein e_μ^a and gravitino ψ_μ^i. The graviphoton is included in the vector multiplets.

- Vector multiplets enumerated by $I = 0, \ldots, n_V$ where n_V is the number of vector multiplets and the fundamental vectors are A_μ^I, and the field strengths $F_{\mu\nu}^I$ are defined with the same normalization as in $d = 5$. We can thus have a gauge algebra with structure constants $f_{JK}{}^I$ and gauge coupling constant g. The fermions of these multiplets are denoted as $\lambda^{\alpha i}$ and the complex scalars as z^α where $\alpha = 1, \ldots, n_V$. The couplings of the vector multiplets is determined by a holomorphic symplectic vector,

$$v(z) = \begin{pmatrix} Z^I(z) \\ F_I(z) \end{pmatrix} , \tag{9.31}$$

such that

$$\langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v \rangle = 2 \left(\mathcal{D}_{[\alpha} Z^I \right) \left(\mathcal{D}_{\beta]} F_I \right) = 0, \qquad \mathcal{D}_\alpha v \equiv \partial_\alpha v + (\partial_\alpha \mathcal{K}) v, \tag{9.32}$$

where the Kähler potential $\mathcal{K}$ is determined by

$$\mathrm{e}^{-\mathcal{K}(z,\bar{z})} = -\mathrm{i}\langle v, \bar{v} \rangle = -\mathrm{i} \left(Z^I \bar{F}_I - F_I \bar{Z}^I \right) \,. \tag{9.33}$$

This Kähler potential gives the metric in the usual way:

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} = \mathrm{i}\mathrm{e}^{\mathcal{K}} \langle \mathcal{D}_\alpha v, \mathcal{D}_{\bar{\beta}} \bar{v} \rangle. \tag{9.34}$$

We require this metric[28] to be positive definite in the physical domain of the scalars z. The ‘usual case’ is when the $(n_V + 1) \times (n_V + 1)$ matrix

$$\begin{pmatrix} Z^I & \mathcal{D}_\alpha Z^I \end{pmatrix} \tag{9.35}$$

[28] There are further cohomological restrictions concerning the global structure of the metric, i.e. it should be Kähler manifolds of restricted type or ‘Hodge manifolds’, but these global restrictions are not discussed here.

is invertible. This can always be obtained by the symplectic transformations of section 6.2 (but one might sometimes prefer not to use such a symplectic basis). In this 'usual case' it can be shown that the condition (6.49) implies that one can define a holomorphic function $F(Z)$, homogeneous of second order in Z, such that

$$F_I(z) = \frac{\partial}{\partial Z^I} F(Z(z)). \tag{9.36}$$

- n_H hypermultiplets with scalars q^X and spinors ζ^A, where $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$. Their interactions are determined by the vielbeins, f_X^{iA}, invertible as $4n_H \times 4n_H$ matrices.

The bosonic action is

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm bos} \;=\;& \tfrac{1}{2\kappa^2}R + \tfrac{1}{4}({\rm Im}\,\mathcal{N}_{IJ})F_{\mu\nu}^I F^{\mu\nu J} - \tfrac{1}{8}({\rm Re}\,\mathcal{N}_{IJ})e^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^I F_{\rho\sigma}^J \\ & - g_{\alpha\bar\beta}\mathcal{D}_\mu z^\alpha \mathcal{D}^\mu \bar z^\beta - \tfrac{1}{2}g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y - V \\ & - \tfrac{1}{6}{\rm i}gC_{I,JK}e^{-1}\varepsilon^{\mu\nu\rho\sigma}A_\mu^I A_\nu^J\left(\partial_\rho A_\sigma^K - \tfrac{3}{8}gf_{LM}{}^K A_\rho^L A_\sigma^M\right). \end{aligned} \tag{9.37}$$

The metric for the vectors is

$$\mathcal{N}_{IJ} \equiv \begin{pmatrix} F_I & \bar{\mathcal{D}}_{\bar\alpha}\bar F_I \end{pmatrix} \begin{pmatrix} Z^J & \bar{\mathcal{D}}_{\bar\alpha}\bar Z^J \end{pmatrix}^{-1}, \tag{9.38}$$

where (for a positive definite metric $g_{\alpha\bar\beta}$), the last matrix is always invertible, and the imaginary part of $\mathcal{N}_{IJ}$ is negative definite. If a prepotential $F(Z)$ exists, this definition leads to

$$\mathcal{N}_{IJ}(z,\bar z) = \bar F_{IJ} + {\rm i}\frac{N_{IN}N_{JK}\,Z^N Z^K}{N_{LM}\;Z^L Z^M}, \qquad N_{IJ} \equiv 2\,{\rm Im}\,F_{IJ} = -{\rm i}F_{IJ} + {\rm i}\bar F_{IJ}, \tag{9.39}$$

where $F_{IJ} = \partial_I \partial_J F$.

The properties of the quaternionic-Kähler manifold and the metric g_{XY} are the same as for $d = 5$.

The potential is generated by the form of the gauge transformations. These transformations act first of all on the matter multiplets (vector multiplet and hypermultiplet). The moment maps determines how also the gravitino transforms, but this is not relevant for this summary. The transformations are very similar to (9.9). This gives

$$\begin{aligned} \delta_G(\alpha)A_\mu^I &= \partial_\mu\alpha^I - g\alpha^J f_{JK}{}^I A_\mu^K, \\ \delta_G v(z) &= g\alpha^I(T_I + r_I)v \\ \delta_G z^\alpha &= -g\alpha^I k_I^\alpha, \qquad k_I^\alpha \equiv -{\rm i}g^{\alpha\bar\beta}\partial_{\bar\beta}P_I^0, \qquad P_I^0 = {\rm e}^{\mathcal{K}}\langle v(z), T_I\bar v(\bar z)\rangle, \\ \delta_G q^X &= -g\alpha^I k_I^X. \end{aligned} \tag{9.40}$$

T_I acts on symplectic sections as follows

$$T_I\begin{pmatrix} Z^J \\ F_J \end{pmatrix} = \begin{pmatrix} -f_{IK}{}^J & 0 \\ C_{I,JK} & f_{IJ}{}^K \end{pmatrix}\begin{pmatrix} Z^K \\ F_K \end{pmatrix}, \tag{9.41}$$

where $C_{I,JK}$ are real coefficients, symmetric in the last two indices, with $Z^I Z^J Z^K C_{I,JK} = 0$.

The potential takes the following form [10]

$$V = g^2 \left(g_{\alpha\bar\beta} k_I^\alpha k_J^{\bar\beta} + 2 g_{XY} k_I^X k_J^Y \right) \mathrm{e}^{\mathcal{K}} \bar Z^I Z^J + 4 g^2 \left(U^{IJ} - 3\mathrm{e}^{\mathcal{K}} \bar Z^I Z^J \right) \vec P_I \cdot \vec P_J, \tag{9.42}$$

where

$$U^{IJ} \equiv g^{\alpha\bar\beta} \mathrm{e}^{\mathcal{K}} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar\beta} \bar Z^J = -\tfrac12 \left(\mathrm{Im}\mathcal{N}\right)^{-1|IJ} - \mathrm{e}^{\mathcal{K}} \overline{Z}^I Z^J. \tag{9.43}$$

It is the sum of three distinct contributions:

$$\begin{aligned} V &= g^2 \left(V_1 + V_2 + V_3\right), \\ V_1 &= g_{\alpha\bar\beta} k_I^\alpha k_J^{\bar\beta} \, \mathrm{e}^{\mathcal{K}} \bar Z^I Z^J, \\ V_2 &= 2 g_{XY} k_I^X k_J^Y \, \mathrm{e}^{\mathcal{K}} \bar Z^I Z^J, \\ V_3 &= 4 \left(U^{IJ} - 3\mathrm{e}^{\mathcal{K}} \bar Z^I Z^J \right) \vec P_I \cdot \vec P_J, \end{aligned} \tag{9.44}$$

V_1 can be rewritten with the help of (6.161), (6.165) and (6.158):

$$\begin{aligned} V_1 = {}& -\mathrm{i}\mathrm{e}^{2\mathcal{K}} \bar Z^I Z^J f_{IJ}{}^K Z^L \left(f_{LK}{}^M \bar F_M + C_{L,KM} \bar Z^M \right) + \text{h.c.} \\ & + \mathrm{e}^{2\mathcal{K}} \left| Z^I \bar Z^J \left(C_{I,JK} Z^K + f_{IJ}{}^K F_K \right) \right|^2 . \end{aligned} \tag{9.45}$$

The second line vanishes if there exist a prepotential (i.e. the Z^I are independent). This is so because this line is then proportional to a gauge transformation of F_I with parameter replaced by Z. We have noticed already that this operation gives zero on X^I, and the F_I in this case depend on X^I.

By their definition, the contributions $V_{1,2}$ are positive definite and the only term that might involve negative contributions is V_3. This can be understood from the fact that V_1 and the first term constitute the square of the supersymmetry transformation of the gauginos (split in the SU(2) triplet and SU(2) singlet part), V_2 is the square of the supersymmetry of the hyperinos, and the last term of V_3 is the square of the gravitino supersymmetry.

The supersymmetry transformations of the fermions to bosons are

$$\begin{aligned} \delta\psi_\mu^i &= D_\mu(\omega)\epsilon^i - g\gamma_\mu S^{ij}\epsilon_j + \tfrac14 \gamma\cdot F^{-I}\varepsilon^{ij}\gamma_\mu\epsilon_j (\mathrm{Im}\,\mathcal{N})_{IJ} Z^J \mathrm{e}^{\mathcal{K}/2}, \\ D_\mu(\omega)\epsilon^i &= \left(\partial_\mu + \tfrac14 \omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon^i + \tfrac12 \mathrm{i} A_\mu \epsilon^i + \partial_\mu q^X \omega_{Xj}{}^i \epsilon^j + g\kappa^2 A_\mu^I P_{Ij}{}^i \epsilon^j, \\ \delta\lambda_i^\alpha &= -\tfrac12 \mathrm{e}^{\mathcal{K}/2} g^{\alpha\bar\beta} \mathcal{D}_{\bar\beta} \bar Z^I (\mathrm{Im}\,\mathcal{N})_{IJ} F_{\mu\nu}^{-J} \gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j + \not{\mathcal{D}} z^\alpha \epsilon_i + g N_{ij}^\alpha \epsilon^j, \\ \delta\zeta^A &= \tfrac12 \mathrm{i} f_X^{Ai} \not{\mathcal{D}} q^X \epsilon_i + g\mathcal{N}^{iA}\varepsilon_{ij}\epsilon^j, \\ S^{ij} &\equiv -P_I^{ij} \mathrm{e}^{\mathcal{K}/2} Z^I, \\ N_{ij}^\alpha &\equiv \mathrm{e}^{\mathcal{K}/2}\left[\varepsilon_{ij} k_I^\alpha \bar Z^I - 2 P_{Iij} \bar{\mathcal{D}}_{\bar\beta} \bar Z^I g^{\alpha\bar\beta} \right], \qquad \mathcal{N}^{iA} \equiv -\mathrm{i} f_X^{iA} k_I^X \mathrm{e}^{\mathcal{K}/2} \bar Z^I \end{aligned} \tag{9.46}$$

(the A index for the hyperinos is to be identified with the $\bar\alpha$ index in previous parts of these lectures, see also [52]). The factor in the gravitino transformation can be rewritten in the case that a prepotential exists as

$$\mathrm{Im}\,\mathcal{N}_{IJ} Z^J = -\mathrm{e}^{-\mathcal{K}} N_{IJ} \bar Z^J \left(\bar Z^K N_{KL} \bar Z^L \right). \tag{9.47}$$

A_μ are the components of the one-form gauge field of the Kähler U(1):

$$A_\mu = -\tfrac{1}{2}\mathrm{i}\left(\partial_\alpha\mathcal{K}\partial_\mu z^\alpha - \partial_{\bar\alpha}\mathcal{K}\partial_\mu \bar z^{\bar\alpha}\right). \qquad (9.48)$$

XXX positivity domain

Exercise 9.1: As a simple example, consider the special Kähler manifold that we have discussed in example 5.2, which has $n = 1$, i.e. 2 vectors. The algebra should be Abelian to leave F invariant. We consider no hypermultiplet, i.e. in the superconformal setup there is just a trivial one that is needed for compensation. Then the potential can only originate in the gauging of the compensating hypermultiplet, which is equivalent to having constant moment maps $\vec P_I$. In the equivariance condition (9.18), the second term is absent as there are no hyperscalars, and the third term as well as the algebra is Abelian. The first term thus implies that $\vec P_1$ and $\vec P_2$ should point in the same direction in SU(2) space. Hence we take

$$g\vec P_I = g_I\vec e\,, \qquad \vec e\cdot\vec e = 1\,. \qquad (9.49)$$

To calculate the potential we thus need first U^{IJ} which can most easily be determined using $\mathcal{D}_z Z^I$. This gives, using (5.53)

$$\begin{aligned}
\mathcal{D}_z Z^I &= \partial_z Z^I + Z^I\partial_z\mathcal{K} = \frac{1}{1-z\bar z}\begin{pmatrix}\bar z\\ 1\end{pmatrix},\\
U^{IJ} &= \frac{1}{1-z\bar z}\begin{pmatrix}z\bar z & \bar z\\ z & 1\end{pmatrix}, \qquad \mathrm{e}^{\mathcal{K}}\bar Z^I Z^J = \frac{1}{1-z\bar z}\begin{pmatrix}1 & \bar z\\ z & z\bar z\end{pmatrix}\\
V &= g^2 V_3 = \frac{4}{1-z\bar z}\left[g_0^2(z\bar z-3) - 2g_0g_1(z+\bar z) + g_1^2(1-3z\bar z)\right].
\end{aligned} \qquad (9.50)$$

Before the gauging the model has an SU(1,1) rigid symmetry. The properties of the potential depend on the SU(1,1)-invariant $g_0^2 - g_1^2$. There are thus 3 relevant cases, whether this invariant is positive, negative or zero [1]. We find respectively the following extrema

- Take $g_0 = g$ and $g_1 = 0$. There is an extremum at $z = 0$ with negative $V(z=0)$, i.e. anti-de Sitter. $g_I U^{IJ} g_J$ vanishes which is the contribution from the supersymmetry of the gaugini. Therefore this vacuum preserves supersymmetry.
- $g_0 = g_1$. In this case there is no extremum in the positivity domain $|z| < 1$.
- $g_0 = 0$ and $g_1 = g$. There is an extremum with positive V, i.e. de Sitter, with nonvanishing $g_I U^{IJ} g_J$, i.e. broken supersymmetry.

Note that in the first case we can omit the scalars, and this is thus pure $N = 2$ supergravity with possible gauging ($g \neq 0$) leading to the anti-de Sitter $N = 2$ supergravity.

10 Further developments

Higher derivatives, see e.g. [119].

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A Notation

I use the metric signature $(-+\ldots+)$. If you prefer the opposite, insert a minus sign for every upper index which you see, or for an explicit metric η_{ab} or $g_{\mu\nu}$. The Gamma matrices γ_a should then be multiplied by an i to have this change of signature.

I use indices as in table 12. Note that it is not possible to have enough symbols to

Table 12: *Use of indices*

μ	$0,\ldots,3$	local spacetime
a	$0,\ldots,3$	tangent spacetime
i	$1,2$	SU(2)
In sections 2 and 3.		
α	$1,\ldots,4$	spinor indices
A		all the gauge transformations
I		all gauge transformations excluding translations
From section 4 onwards.		
I	$0,\ldots,n$	vector multiplets
X	$1,\ldots,4r$	scalars in hypermultiplets in 5 dim.
A	$1,\ldots,2r$	spinors (or $USp(2r)$ vector) in hypermultiplets
r	$1,2,3$	triplet of $SU(2)$
α	$1,\ldots,n$	independent coordinates in special Kähler

indicate all indices and numbers with different symbols. Indices and numbers should be distinguished. An index occurs never as a number. E.g. when r appears as a factor in a formula, it is the number of hypermultiplets, and is independent of the index r.

(Anti)symmetrization is done with weight one:

$$A_{[ab]}=\tfrac12\left(A_{ab}-A_{ba}\right)\qquad\text{and}\qquad A_{(ab)}=\tfrac12\left(A_{ab}+A_{ba}\right)\,. \tag{A.1}$$

The antisymmetric tensors are often contracted with γ matrices as in $\gamma\cdot T\equiv\gamma^{ab}T_{ab}$.

A.1 Bosonic part

For the curvatures and connections, I use the conventions:

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab}(e)+2\omega_{[\mu}{}^{ac}(e)\omega_{\nu]c}{}^{b}(e)\,,\\
R^{\mu}{}_{\nu\rho\sigma} &= R_{\rho\sigma}{}^{ab}e_a^{\mu}e_{\nu b}=2\partial_{[\rho}\Gamma^{\mu}_{\sigma]\nu}+2\Gamma^{\mu}_{\tau[\rho}\Gamma^{\tau}_{\sigma]\nu}\,,\\
R_{\mu\nu} &= R_{\rho\mu}{}^{ba}e_b{}^{\rho}e_{\nu a}=R^{\rho}{}_{\nu\rho\mu}\,,\qquad R=g^{\mu\nu}R_{\mu\nu}\,,\\
G_{\mu\nu} &= e^{-1}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\,eR=R_{\mu\nu}-\tfrac12 g_{\mu\nu}R\,.
\end{aligned} \tag{A.2}$$

Note that the above signs define the $(+++)$ convention in terms of the classification of conventions found on the first page of [120]. These signs concern the signature convention – curvature definition – Ricci tensor definition, see also section C.1.

The formulations in terms of spin connection ω and in terms of Levi-Civita connection Γ are equivalent by demanding

$$0 = \nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab}(e) e_{\nu b} - \Gamma^\rho_{\mu\nu} e_\rho{}^a \,, \qquad g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b \,, \tag{A.3}$$

which leads to

$$\begin{aligned} \omega_\mu{}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma{}^c \,, \\ \Gamma^\rho_{\mu\nu} &= \tfrac12 g^{\rho\lambda}\left(2\partial_{(\mu} g_{\nu)\lambda} - \partial_\lambda g_{\mu\nu}\right) \,, \qquad \Gamma^\nu_{\mu\nu} = \tfrac12 \partial_\nu \ln g \,. \end{aligned} \tag{A.4}$$

Note that the Ricci tensor and scalar curvature are of opposite sign of many of my previous papers. The sign is now chosen such that Einstein spaces with positive scalar curvature are compact. Another useful formula to show the conventions is for any V_ρ

$$[\nabla_\mu, \nabla_\nu]\, V_\rho = -R^\sigma{}_{\rho\mu\nu} V_\sigma . \tag{A.5}$$

The anticommuting Levi–Civita tensor is real, and taken to be

$$\varepsilon_{0123} = 1 \,, \qquad \varepsilon^{0123} = -1 \,, \tag{A.6}$$

the $-$ sign is of course related to the one timelike direction[29]. For convenience, I give below the formulae for an arbitrary dimension d and number of timelike directions t. The contraction identity for these tensors is $(p+n=d)$

$$\varepsilon_{a_1\ldots a_n b_1 \ldots b_p} \varepsilon^{a_1\ldots a_n c_1 \ldots c_p} = (-)^t\, p!\, n!\, \delta^{[c_1}_{[b_1} \ldots \delta^{c_p]}_{b_p]} \,. \tag{A.7}$$

For the local case, we can still define constant tensors

$$\varepsilon_{\mu_1\ldots\mu_d} = e^{-1} e^{a_1}_{\mu_1} \ldots e^{a_d}_{\mu_d} \varepsilon_{a_1\ldots a_d} \,, \qquad \varepsilon^{\mu_1\ldots\mu_d} = e e^{\mu_1}_{a_1} \ldots e^{\mu_d}_{a_d} \varepsilon^{a_1\ldots a_d} \,. \tag{A.8}$$

They are thus *not* obtained from each other by raising or lowering indices with the metric. The $\varepsilon_{\mu_1\cdots\mu_d}$ correspond to the same symbol in the lectures of C. Pope XXX. The $\epsilon_{\mu_1\cdots\mu_d}$ that he introduces correspond to the $\varepsilon_{a_1\cdots a_d}$ with indices changed to local indices.

Exercise A.1: Show that the tensors in (A.8) are indeed constants, i.e. that arbitrary variations of the vierbein cancel in the full expression. You can use the so-called 'Schouten identities', which means that antisymmetrizing in more indices than the range of the indices, gives zero. The constancy thus implies that one can have (A.6) without specifying whether the 0123 are local or flat indices.

[29] Note that in many papers, e.g. [35], the lectures in this institute of Peter van Nieuwenhuizen, and many papers of myself, one takes in 4 dimensions an imaginary Levi–Civita tensor to avoid factors of i in definitions of duals, ...

Exercise A.2: If one defines in even dimensions $d = 2n$ the dual of an n-tensor as

$$\tilde{F}_{a_1\ldots a_n} = -(\mathrm{i})^{d/2+t}\frac{1}{n!}\varepsilon_{a_1\ldots a_d}F^{a_d\ldots a_{n+1}}\,, \tag{A.9}$$

check that

$$\tilde{\tilde{F}}_{a_1\ldots a_n} = F_{a_1\ldots a_n}. \tag{A.10}$$

The self-dual and anti-self-dual tensors are introduced in even dimensions as

$$F^{\pm}_{a_1\ldots a_n} \equiv \tfrac{1}{2}\left(F_{a_1\ldots a_n} \pm \tilde{F}_{a_1\ldots a_n}\right)\,. \tag{A.11}$$

The definition (A.9) is in 4 and 6 dimensions respectively

$$d=4\ :\ \tilde{F}^{ab} \equiv -\tfrac{1}{2}\mathrm{i}\varepsilon^{abcd}F_{cd}\,,\qquad d=6\ :\ \tilde{F}^{abc} \equiv \tfrac{1}{3!}\varepsilon^{abcdef}F_{def}\,. \tag{A.12}$$

The minus sign in the definition of the dual is convenient for historical reasons. Indeed, when, as written in footnote 29, this ε is i times the ε in these earlier papers then this agrees with the operation that was taken there (e.g. in [35]). In 4 dimensions the dual is an imaginary operation, and the complex conjugate of a self-dual tensor is its anti-self-dual partner, while in 6 dimensions the (anti)-self-dual part of a real tensor is real.

It is useful to observe relations between (anti)self-dual tensors. In 4 dimensions there are

$$\begin{aligned} &G^{+ab}H^{-}{}_{ab} = 0\,, \qquad G^{\pm c(a}H^{\pm b)}{}_{c} = -\tfrac{1}{4}\eta^{ab}G^{\pm cd}H^{\pm}{}_{cd}\,,\\ &G^{+}_{c[a}H^{-c}_{b]} = 0\,. \end{aligned} \tag{A.13}$$

In 6 dimensions

$$\begin{aligned} &G^{\pm abc}H^{\pm}{}_{abc} = 0\,, \qquad G^{+cd(a}H^{-b)}{}_{cd} = \tfrac{1}{6}\eta^{ab}G^{+cde}H^{-}{}_{cde}\,,\\ &G^{\pm}_{cd[a}H^{\pm}{}_{b]}{}^{cd} = 0\,, \qquad G^{\pm}_{a[bc}H^{\pm}{}_{cd]}{}^{a} = 0\,. \end{aligned} \tag{A.14}$$

For $N = 2$ we use the Levi–Civita ε^{ij} for which the important property is that

$$\varepsilon_{ij}\varepsilon^{jk} = -\delta_i{}^k\,, \tag{A.15}$$

where in principle ε^{ij} is the complex conjugate of ε_{ij}, but we can use ($\varepsilon = \mathrm{i}\sigma_2$)

$$\varepsilon_{12} = \varepsilon^{12} = 1\,. \tag{A.16}$$

Note how I wrote the position of the indices on the delta function in (A.15). If you remember that a delta function has this position of indices, then δ and ε can be seen as the same tensor with indices raised or lowered, i.e. $\delta_i{}^j = \varepsilon_i{}^j$. Switching the positions of contracted SU(2) indices leads to a minus sign:

$$A^i{}_i = -A_i{}^i\,. \tag{A.17}$$

Another useful relation is that for any antisymmetric tensor in ij:

$$A^{[ij]} = -\tfrac{1}{2}\varepsilon^{ij}A^k{}_k\,. \tag{A.18}$$

A.2 Gamma matrices and spinors

A general treatment of gamma matrices and spinors is given in section 3 of [121]. In that review general spacetime signatures are treated. Of course, that material is not original, and is rather a convenient reformulation of earlier works [122, 123, 124, 125]. Another approach to the theory of spinors has been presented in [26]. We review here the essential properties for the dimensions used in this work.

A.2.1 $d = 4$

Gamma matrices and chirality. We do not need the explicit representations of gamma matrices. Three different representations are given in section 3.3 of [121], but the only equations that we need are

$$\begin{array}{l} \gamma_a\gamma_b = \eta_{ab} + \gamma_{ab}\,, \qquad \gamma_5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \\ \gamma_5^2 = 1\,, \qquad \gamma_5\gamma_a = \frac{1}{3!}\mathrm{i}\varepsilon_{abcd}\gamma^{bcd}\,, \qquad \varepsilon_{abcd}\gamma^d = \mathrm{i}\gamma_5\gamma_{abc}, \qquad \gamma_5\gamma_{ab} = -\tilde{\gamma}_{ab}\,. \end{array} \tag{A.19}$$

In the first expression is the antisymmetric γ_{ab}, which is often denoted as $2\sigma_{ab}$. I avoid this notation, as it is special for the 4-dimensional case. I use left and right projections

$$P_L = \tfrac{1}{2}(1+\gamma_5)\,, \qquad P_R = \tfrac{1}{2}(1-\gamma_5)\,. \tag{A.20}$$

In four dimensions, one often uses the place of an index, e.g. the $i = 1, 2$ index for doublets of $N = 2$ supersymmetry, to denote chiral spinors. Thus, e.g. for the supersymmetry parameter one uses

$$\epsilon^i = P_L\epsilon^i\,, \qquad \epsilon_i = P_R\epsilon_i\,. \tag{A.21}$$

Thus ϵ^i is not a Majorana spinor, but a chiral spinor. The choice of whether the upper index denotes the left or the right chiral spinor is indicated when such a spinor is introduced. Thus, for some chiral spinors, the upper index denotes the right-chiral one rather than the left-chiral one as here for ϵ^i.

One representation of the gamma matrices allows to connect easily to the papers that use a 2-component notation, see e.g. the review of supergravity couplings [8]. This is a representation in which γ_5 is diagonal:

$$\gamma_0 = \begin{pmatrix} 0 & \mathrm{i}\mathbb{1}_2 \\ \mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & -\mathrm{i}\vec{\sigma} \\ \mathrm{i}\vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{A.22}$$

For convenience, let me repeat that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.23}$$

The only relevant properties are $\sigma_1\sigma_2 = \mathrm{i}\sigma_3$ and cyclic, and that they square to $\mathbb{1}_2$ and are Hermitian. With the choice (A.22), we have

$$\mathcal{C} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad \text{with} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.24}$$

Reality and charge conjugation About the *reality conditions*, I first give the definitions, but, for all practical purposes one can suffice by replacing complex conjugation with a charge conjugation operation, as I will explain subsequently.

For the definitions, γ_0 is anti-Hermitian, while the spacelike components of γ^μ and γ_5 are Hermitian. The spinors are Majorana, which means that

$$\bar\lambda \equiv \lambda^T\mathcal{C} = \lambda^\dagger\gamma_0\alpha^{-1} \qquad \text{or} \qquad \lambda^* = -\alpha\mathcal{C}\gamma_0\lambda\,, \tag{A.25}$$

where $\mathcal{C}$ is the (unitary and antisymmetric) charge conjugation matrix and α is a number with modulus 1.

One can arbitrary choose whether complex conjugation reverses the order of spinors. . This is indicated by a number β:

$$(\lambda\chi)^* = \beta\lambda^*\chi^* = -\beta\chi^*\lambda^*\,, \qquad \beta = \pm 1\,. \tag{A.26}$$

This finishes the definitions. Below, I relate the choice of α to the choice of β, such that $\beta\alpha^2 = 1$. Then I leave it to your personal preference to choose β and α, as this is immaterial for everything that follows. Remark that in many papers you will see for all bilinears of fermions an extra factor of i with respect to what I write. This is then due to a choice of α and β such that $\beta\alpha^2 = -1$.

Now the practical way [121]. When one writes +h.c., you consider instead the operation called charge conjugation (C). E.g. for spinors it is defined by

$$\lambda^C = \alpha^{-1}\gamma_0\mathcal{C}^{-1}\lambda^*. \tag{A.27}$$

Majorana spinors are according to (A.25) those spinors that are real under this conjugation. For all practical purposes you do not need this explicit definition. The operation of charge conjugation leaves the order of the spinors unchanged, independent of the choice of β, i.e. whether complex conjugation interchanges the order of the fermions. For bosons, the charge conjugation is just complex conjugation, and thus e.g. changes in 4 dimensions the self-dual in the anti-self-dual, as remarked after (A.12). For the gamma matrices one uses the following. For matrices in spinor space, we have $\gamma_\mu^C = \gamma_\mu$ and $\gamma_5^C = -\gamma_5$. Thus, only γ_5 should be considered as pure imaginary. This implies that the left and right projections in (A.20) are interchanged. Thus, chiral spinors are replaced by anti-chiral ones and vice-versa. The same holds for the barred (Majorana conjugate) spinors. In our notation where the position of the index i denotes the chirality, the operation thus interchanges upper with lower i indices.

One more useful information is that the Majorana conjugate of a chiral spinor works as follows[30]

$$\overline{P_L\chi} = \bar\chi P_L\,, \qquad \bar\chi^i \equiv \overline{\chi^i}\,. \tag{A.28}$$

[30]Observe that the bar denotes always a Majorana conjugate, not a Dirac conjugate. As such, the first equality in (A.25) is the definition, and the second equality is what holds for Majorana spinors, but not for general spinors, i.e. not for chiral spinors, which cannot be Majorana in 4 dimensions

As a practical summary of the rules, see the following equations:

$$\begin{aligned}&\gamma_a^C = \gamma_a\,, \qquad \gamma_5^C = -\gamma_5\,, \qquad P_L^C = P_R\,,\\ &\lambda^C = \lambda\,, \qquad \bar\lambda^C = \bar\lambda\,, \qquad \chi_i^C = \chi^i\,, \qquad \bar\chi_i^C = \bar\chi^i\,,\\ &\text{h.c.}\left(\bar\lambda\psi\right) = \bar\lambda\psi\,, \qquad \text{h.c.}\left(\bar\lambda\gamma_5\psi\right) = -\bar\lambda\gamma_5\psi\,, \qquad \text{h.c.}\left(\bar\chi^i\gamma_a\phi_i\right) = \bar\chi_i\gamma_a\phi^i\,. \end{aligned} \tag{A.29}$$

Here and elsewhere in the text, we assume that spinors like λ and ψ are Majorana, and spinors with an index, like χ^i and ϕ^i are chiral.

Exercise A.3: Suppose that a Majorana spinor λ is given by its left-chiral part, which is $P_L\lambda = m_i\chi^i$ where m_i is a complex number, and χ_i a (left) chiral spinor doublet. Show that the appropriate definition for m^i is $m^i \equiv (m_i)^*$ and that $\lambda = m_i\chi^i + m^i\chi_i$. Then suppose that $\chi^i = n^iP_L\psi$, with m_in^i pure imaginary, $m_in^i = \mathrm{i}p$ with p real, and ψ Majorana. Then $\lambda = \mathrm{i}\gamma_5\psi$. Check directly the C-invariance of this equation.

We finish with a summary in a form to compare below with $d = 5$ and $d = 6$.

- *a bispinor* $\bar\lambda\psi$ *is real.*
- $\gamma_a^C = \gamma_a$, and $\gamma_5^C = -\gamma_5$
- For a bosonic matrix in SU(2) space, the charge conjugation changes the height of the indices, e.g. $(M^{ij})^C = M_{ij}$, or $(M^i{}_j)^C = M_i{}^j$.
- Charge conjugation changes also the position of the index i on spinors. This position indicates the chirality, see table 3 and table 4. For the fermions of hypermultiplets, the index α plays the same role, i.e. its position is changed by the C operation.

For hypermultiplets, we can add that we also define the charge conjugate of ζ^A as ζ_A, changing its chirality. The charge conjugation thus also changes the height of the symplectic index A and similarly, see (4.60):

$$(f_X^{iA})^C = (f_X^{iA})^* = f_X^{jB}\varepsilon_{ji}\rho_{BA} = f_{XiA}. \tag{A.30}$$

For the USp($2r$) connection one has

$$(\omega_{XA}{}^B)^C = (\omega_{XA}{}^B)^* = -\rho^{AD}\omega_{XD}{}^E\rho_{EB} = -\omega_X{}^A{}_B, \tag{A.31}$$

where we raise and lower indices on the bosonic quantities with ρ (we can not do this here with fermions, as the position of the index indicates their chirality, and thus we should not raise or lower it with a matrix that does not change the chirality). Therefore $\omega_{XA}{}^B$ is in a way pure imaginary under charge conjugation similar to the γ_5 matrix. This means that apart from raising and lowering the indices, we have to insert a minus sign when taking charge conjugates. Note that the last expression can also be written as $-\omega_{XB}{}^A$ due to (4.89).

SU(2) **and vector indices.** At various places in the main text, we switch from SU(2) indices $i, j = 1, 2$ to vector quantities with the convention

$$A_i{}^j \equiv \mathrm{i}\vec{A} \cdot \vec{\sigma}_i{}^j \,, \tag{A.32}$$

where $\vec{\sigma}_i{}^j$ are the Pauli matrices (A.23). With these conventions, we obtain the identity

$$A_i{}^j B_j{}^k = -\vec{A} \cdot \vec{B} \delta_i{}^k - \mathrm{i}(\vec{A} \times \vec{B}) \cdot \vec{\sigma}_i{}^k \,, \qquad \text{i.e.} \;\; \vec{A} \cdot \vec{B} = -\tfrac{1}{2} A_i{}^j B_j{}^i = \tfrac{1}{2} A^{ij} B_{ji} \,, \tag{A.33}$$

for any two vectors $\vec{A}$ and $\vec{B}$. We have also various symmetric tensors like Y_{ij} that satisfy

$$Y_{ij} = -\varepsilon_{ik} Y^{k\ell} \epsilon_{\ell j} \,, \qquad \text{where } Y^{ij} = Y_{ij}^* \,, \tag{A.34}$$

These are thus the symmetric tensors that are invariant under the charge conjugation. Defining the 3 matrices

$$\vec{\sigma}_{ij} = \vec{\sigma}_i{}^k \varepsilon_{kj} \,, \tag{A.35}$$

we can also write

$$Y_{ij} = \mathrm{i}\, \vec{Y} \cdot \vec{\sigma}_{ij} \,, \qquad Y_{ij} Y^{ij} = 2\, \vec{Y} \cdot \vec{Y} \,. \tag{A.36}$$

A.2.2 $d = 5$

Gamma matrices and SU(2) **indices.** The essential new property for the gamma matrices is that the antisymmetric product of all the matrices is proportional to the unit matrix:

$$\gamma^{abcde} = \mathrm{i}\varepsilon^{abcde} \,. \tag{A.37}$$

Exercise A.4: Check that this implies

$$\begin{aligned}
\gamma^{abcd} &= \mathrm{i}\varepsilon^{abcde}\gamma_e \,, \\
2\gamma^{abc} &= \mathrm{i}\varepsilon^{abcde}\gamma_{ed} \,, \\
3!\gamma^{ab} &= \mathrm{i}\varepsilon^{abcde}\gamma_{edc} \,, \\
4!\gamma^{a} &= \mathrm{i}\varepsilon^{abcde}\gamma_{edcb} \,, \\
5! &= \mathrm{i}\varepsilon^{abcde}\gamma_{edcba} \,, \\
\mathrm{i}\varepsilon^{abcde}\gamma_{ef} &= 4\gamma^{[abc}\delta_f^{d]} \,, \\
\mathrm{i}\varepsilon^{abcde}\gamma_{efg} &= 12\gamma^{[ab}\delta_{gf}^{cd]} \,.
\end{aligned} \tag{A.38}$$

In odd dimensions there is no chirality. The position of the indices i on the spinors is thus not used to indicate chirality. Rather these can now be raised or lowered using ε^{ij}. I use NorthWest–SouthEast (NW–SE) convention, which means that this is the direction in which contracted indices should appear to raise or lower indices: see

$$X^i = \varepsilon^{ij} X_j \,, \qquad X_i = X^j \varepsilon_{ji} \,. \tag{A.39}$$

Exercise A.5: Check that we can consider ε_{ij} as the tensor $\delta_i{}^j$ with the j-index lowered. Note that it is important then to write $\delta_i{}^j$ and not $\delta^j{}_i$. Also ε^{ij} is the corresponding tensor with raised indices.

Note that we cannot do this in 4 dimensions, as we attached there a meaning to the position of the index i related to chirality, and interchanged by complex conjugation. Implicit summation (in NW–SE direction) is also used for bilinears of fermions, e.g.

$$\bar\lambda\chi \equiv \bar\lambda^i\chi_i\,. \tag{A.40}$$

The transition between SU(2) indices $i,j = 1,2$ and vector quantities as explained for 4 dimensions after (A.32) applies also here and relations as (A.34) are then consistent with the raising and lowering of indices as in (A.39).

Reality and charge conjugation In this case, the charge conjugation $\mathcal{C}$ and $\mathcal{C}\gamma_a$ are antisymmetric. In this case the elementary spinors are 'symplectic Majorana', which means that

$$(\lambda^i)^C \equiv \alpha^{-1}\gamma_0\mathcal{C}^{-1}(\lambda^j)^*\varepsilon^{ji}, \tag{A.41}$$

where again α is a number of modulus one, and we assume further that $\beta\alpha^2 = 1$, for β defining whether fermions change order under complex conjugation as in (A.26). Let us go directly to the practical rules for replacing complex conjugation by charge conjugation. Then symplectic Majorana spinors can be considered as real, and for these rules we do not have to change the order of the spinors or gamma matrices. However there are some differences with the 4-dimensional rules:

- *a bispinor $\bar\lambda^i\chi_i$ is pure imaginary.*
- $\gamma_a^C = -\gamma_a$.
- For a bosonic matrix in SU(2) space, M, we have $M^C = \sigma_2 M^*\sigma_2$, or explicitly $(M_i{}^j)^C = -\epsilon_{ik}(M_k{}^\ell)^*\varepsilon^{\ell j}$.
- Charge conjugation does not change the position of the index i. This position can be changed by multiplying with ε^{ij}.

As an example, see that the expression

$$\bar\lambda^i\gamma^\mu\partial_\mu\lambda_i \tag{A.42}$$

is real for symplectic Majorana spinors.

For hypermultiplets the reality condition depends on a matrix ρ_{AB}, which is introduced in (4.23). The charge conjugation under which the symplectic Majorana spinors are invariant is then similar to (A.41):

$$(\zeta^A)^C \equiv \alpha^{-1}\gamma_0\mathcal{C}^{-1}(\zeta^B)^*\rho^{BA} = \zeta^A. \tag{A.43}$$

The generalization of the relation for a bosonic matrix in SU(2) space now includes multiplication with iρ matrices. This then e.g. implies that f_X^{iA} is imaginary under 5-dimensional charge conjugation, contrary to the 4-dimensional charge conjugation. For other objects, e.g. $\omega_{XA}{}^B$, one uses as charge conjugate

$$(\omega_{XA}{}^B)^C = (-\rho_{AD})(\omega_{XD}{}^E)^*\rho^{EB} = \omega_{XA}{}^B, \tag{A.44}$$

where the first equation shows the procedure for any lower or upper index, and the last equation is just true because of the particular reality condition of the USp($2r$) connection.

A.2.3 $d = 6$

Gamma matrices and chirality. Here there is again chirality, as for every even dimension, but moreover, as we have seen already there are real self-dual tensors. Similarly, we can define

$$\gamma_7 = \gamma^0 \dots \gamma^5 = -\gamma_0 \dots \gamma_5\,, \tag{A.45}$$

without a factor i. The essential formula is

$$\gamma_{abc}\gamma_7 = -\tilde{\gamma}_{abc}\,, \tag{A.46}$$

Exercise A.6: Show that $\gamma_{abc}P_L = \gamma^-_{abc}$ as in 4 dimensions where $\gamma_{ab}P_L = \gamma^-_{ab}$.

Spinors can satisfy the (symplectic) Majorana condition and be chiral at the same time. Thus complex conjugation does not change chirality, and we can raise and lower indices as in (A.39).

Reality and charge conjugation We choose the charge conjugation matrix to be symmetric[31]. Then the rules for charge conjugation are

- *a bispinor* $\bar\lambda^i\chi_i$ *is real.*
- $\gamma_a^C = \gamma_a$ and $\gamma_7^C = \gamma_7$.
- For a bosonic matrix in SU(2) space, M, we have $M^C = \sigma_2 M^* \sigma_2$.
- Charge conjugation does not change the position of the index i. This position can be changed by multiplying with ε^{ij}.

A.3 Gamma matrix manipulations

In this section, I write some general identities for manipulations of gamma matrices. It is as easy to write this in general dimensions, and thus this section is not restricted to $d = 4, 5, 6$. To indicate this clearly, I use Γ rather than γ for formulas that have a more general validity.

[31] That is a choice in 6 dimensions, as we can also use the antisymmetric charge conjugation matrix $\mathcal{C}' = \mathcal{C}\gamma_7$.

A.3.1 Spinor indices

I mostly omit spinor indices. However, to make the connection with the abstract algebra, it is useful to introduce them. I will give here a way to include spinor indices in a practical way, independent of the dimension of spacetime. First of all, spinors get a lower spinor index. So for a spinor λ, I write λ_α. Gamma matrices act on these spinors, and therefore the expression $\Gamma_a\lambda$ becomes $(\Gamma_a)_\alpha{}^\beta\lambda_\beta$. Note the position of the spinor indices on a usual Gamma matrix. This is then the same for products of Gamma matrices. A Majorana conjugated spinor $\bar\lambda$ is written as λ^α. Thus, see the translation:

$$\bar\lambda\chi \to \lambda^\alpha\chi_\alpha\,, \qquad \bar\lambda\Gamma_a\chi \to \lambda^\alpha(\Gamma_a)_\alpha{}^\beta\chi_\beta\,. \tag{A.47}$$

Then, I want to be able to raise and lower indices. Indeed, e.g. to discuss symmetries of Gamma matrices, the $(\Gamma_a)_\alpha{}^\beta$ cannot be used, as we cannot interchange upper and lower indices. To that purpose, I introduce matrices $\mathcal{C}^{\alpha\beta}$ and $\mathcal{C}_{\alpha\beta}$, which will be related to the charge conjugation matrix below. The convention that I adopt, is that I raise and lower indices always in the NW–SE convention. That means that the contraction indices should appear in that relative position, i.e.

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta\,, \qquad \lambda_\alpha = \lambda^\beta\mathcal{C}_{\beta\alpha}\,. \tag{A.48}$$

In order for these two equations to be consistent, we should have

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha\,, \qquad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma\,. \tag{A.49}$$

This is thus the same as for the SU(2) conventions mentioned in section A.1, and again it is good to remember that a delta function is a charge conjugation with indices in the position as in $\delta_\alpha{}^\beta = \mathcal{C}_\alpha{}^\beta$.

Comparing (A.48) and (A.25), we conclude that $\mathcal{C}^{\alpha\beta}$ is $\mathcal{C}^T$, and $\mathcal{C}_{\alpha\beta}$ is $\mathcal{C}^{-1}$. The symmetry of the charge conjugation matrix determines what happens when we raise and lower indices. Define t_0 by

$$\mathcal{C}_{\alpha\beta} = -t_0\mathcal{C}_{\beta\alpha}\,. \tag{A.50}$$

This t_0 is thus the parameter ϵ in section 3 of [121], and you find it in table 13. Then raising and lowering an index goes as

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha\,. \tag{A.51}$$

Thus we have + for 6 dimensions, but − for 4 and 5 dimensions. The same is true for spinor indices at any place: $(\Gamma_a)_\alpha{}^\beta\lambda_\beta = \pm(\Gamma_a)_{\alpha\beta}\lambda^\beta$.

Denoting for short $\Gamma^{(n)}$ for an antisymmetric product of n gamma matrices:

$$\Gamma^{(n)} = \Gamma_{[a_1}\Gamma_{a_2}\ldots\Gamma_{a_n]}\,, \tag{A.52}$$

the symmetry of $\Gamma^{(n)}_{\alpha\beta} \equiv \Gamma^{(n)}_\alpha{}^\Gamma\mathcal{C}_{\Gamma\beta}$ is given by

$$\Gamma^{(n)}_{\alpha\beta} = -t_n\Gamma^{(n)}_{\beta\alpha}\,, \tag{A.53}$$

where the t_n are the sign factors in table 13.

Table 13: *Symmetries of gamma matrices: sign factors t_n. The values of n with $t_n = +1$ and $t_n = -1$ are modulo 4. For even dimensions there are two possibilities. This is related to changing chiralities of spinors. For d $= 0, 4, 8, 12$ only the first line is used in supersymmetry, as this allows the minimal supersymmetry algebras. In d $= 2, 6, 10$ it does not really matter what you use, as the two choices are related by a redefinition of the charge conjugation. The sign changes that this produces only appears when the two spinors are of opposite chirality, and thus this is not a relevant sign. Note that the numbers appearing in the column $t_n = -1$ also indicate in how many time directions (modulo 4) Majorana spinors can be defined for the corresponding dimension and charge conjugation. The fact that '1' (Minkowski space) does not appear for d $= 5$ and d $= 6$ is the reason why we need symplectic Majorana spinors. The cases that we use in this review are indicated in boldface.*

d (mod 8)	$t_n = -1$	$t_n = +1$
0	0,3	2,1
	0,1	2,3
1	0,1	2,3
2	0,1	2,3
	1,2	0,3
3	1,2	0,3
4	**1,2**	**0,3**
	2,3	0,1
5	**2,3**	**0,1**
6	2,3	0,1
	0,3	**1,2**
7	0,3	1,2

A.3.2 Symmetries of bilinears.

For bilinears of fermions, an extra sign comes in due to the interchange of the fermions. We thus have

$$\bar\lambda\Gamma^{(n)}\chi = t_n\bar\chi\Gamma^{(n)}\lambda\,. \tag{A.54}$$

Note that in this section we are not concerned with SU(2) *indices in 5 and 6 dimensions. Thus, before using this section in these dimensions, you reinstall the indices explicitly if they are not written, and keep them attached to the spinors* Therefore, in $d=5$ and $d=6$, then there is an extra $-$ sign as e.g. in anticommutators of two supersymmetries

$$\bar\epsilon_2\gamma_a\epsilon_1 = \bar\epsilon_2^i\gamma_a\epsilon_{1i} = \bar\epsilon_{1i}\gamma_a\epsilon_2^i = -\bar\epsilon_1^i\gamma_a\epsilon_{2i} = -\bar\epsilon_1\gamma_a\epsilon_2\,. \tag{A.55}$$

The second equality uses (A.54) with $t_1=1$. The minus sign of (A.17) produces then the sign in the next equality.

The same sign factors can be used in a longer chain of gamma matrices:

$$\bar\lambda\Gamma^{(n_1)}\Gamma^{(n_2)}\cdots\Gamma^{(n_p)}\chi = t_0^{p-1}t_{n_1}t_{n_2}\cdots t_{n_p}\,\bar\chi\gamma^{(n_p)}\cdots\gamma^{(n_2)}\gamma^{(n_1)}\lambda\,. \tag{A.56}$$

We thus just multiply the sign factors of the gamma matrices and transpose them, but in dimension where $\mathcal{C}$ is symmetric ($t_0=-1$), there is an extra sign factor. For us, this happens in 6 dimensions.

As an example, consider in 4 dimensions (γ_5 is an antisymmetric product of type $\gamma^{(4)}$)

$$\bar\lambda\gamma_a\gamma_{bc}\xi = \bar\chi\gamma_{bc}\gamma_a\lambda\,,\qquad \bar\lambda\gamma_a\gamma_5\xi = -\bar\chi\gamma_5\gamma_a\lambda\,. \tag{A.57}$$

Exercise A.7: Check that due to $t_0=t_4$ in 4 dimensions, and $t_0=-t_6$ in 6 dimensions, a bilinear of two chiral spinors of the same chirality should have an even number of γ matrices in 4 dimensions. E.g. $\bar\xi_L\phi_L\neq 0$, while it should have an odd number in 6 dimensions (the former example would be zero while $\bar\xi_L\gamma_{abc}\phi_L\neq 0$).

Exercise A.8: Check with the rules of raising and lowering indices that if

$$\lambda = \Gamma^{(n_1)}\Gamma^{(n_2)}\cdots\Gamma^{(n_p)}\chi\ \longrightarrow\ \bar\lambda = t_0^p\,t_{n_1}t_{n_2}\cdots t_{n_p}\,\bar\chi\Gamma^{(n_p)}\cdots\Gamma^{(n_2)}\Gamma^{(n_1)}\,. \tag{A.58}$$

A.3.3 Products of Γ matrices and Fierzing

There are some useful identities for calculations in arbitrary dimensions [25]. For a product of two antisymmetrized gamma matrices, one can use

$$\Gamma_{a_1\ldots a_i}\Gamma^{b_1\ldots b_j} = \sum_{k=|i-j|}^{i+j}\frac{i!j!}{s!t!u!}\delta^{[b_1}_{[a_i}\cdots\delta^{b_s}_{a_{t+1}}\Gamma_{a_1\ldots a_t]}{}^{b_{s+1}\ldots b_j]} \tag{A.59}$$
$$s=\tfrac12(i+j-k)\,,\qquad t=\tfrac12(i-j+k)\,,\qquad u=\tfrac12(-i+j+k)\,.$$

The numeric factor can be understood as follows. Between the i indices of the first Gamma factor, I select $s=i-t$ for the contraction. That choice is a factor $\binom{i}{s}$. The same number of indices s is chosen between the j indices of the second factor. That is a factor $\binom{j}{s}$. Finally, I can contract these s indices in $s!$ ways. In [25] a few extra rules are given and a diagrammatic technique is explained that is based on the work of Kennedy [126].

For contractions of repeated gamma matrices, one has the formula

$$\begin{aligned}\Gamma_{b_1\ldots b_k}\Gamma_{a_1\ldots a_\ell}\Gamma^{b_1\ldots b_k} &= c_{k,\ell}\Gamma_{a_1\ldots a_\ell}\\ c_{k,\ell} &= (-)^{k(k-1)/2}k!(-)^{k\ell}\sum_{i=0}^{\min(k,\ell)}\binom{\ell}{i}\binom{D-\ell}{k-i}(-)^i\,,\end{aligned}\tag{A.60}$$

for which tables were given in [25] in dimensions 4, 10, 11 and 12, and which can be easily obtained from a computer programme. Useful examples in 4 dimensions are

$$\gamma^a\gamma^{bc}\gamma_a=\gamma^{ab}\gamma_c\gamma_{ab}=0,\qquad \gamma^{ab}\gamma_{cd}\gamma_{ab}=4\gamma_{cd}.\tag{A.61}$$

Further, there is the Fierz relation. We know that the gamma matrices are matrices in dimension $\Delta=2^{\mathrm{Int}\ d/2}$, and that a basis of $\Delta\times\Delta$ matrices is given by the set

$$\{\mathbb{1},\Gamma_a,\Gamma_{a_1a_2},\ldots,\Gamma^{a_1\ldots a_{[D]}}\}\qquad\text{where}\qquad\left\{\begin{array}{ll}[D]=D & \text{for even } D\\ {[D]}=(D-1)/2 & \text{for odd } D\,,\end{array}\right.\tag{A.62}$$

of which only the first has nonzero trace. This is the basis of the general Fierz formula for an arbitrary matrix M in spinor space:

$$2^{\mathrm{Int}(d/2)}M_\alpha{}^\beta=\sum_{k=0}^{[D]}(-)^{k(k-1)/2}\tfrac{1}{k!}\left(\Gamma_{a_1\ldots a_k}\right)_\alpha{}^\beta\ \mathrm{Tr}\left(\Gamma^{a_1\ldots a_k}M\right)\,.\tag{A.63}$$

Further Fierz identities can be found in [126].

Exercise A.9: Check that Fierz identities for chiral spinors in 4 dimensions lead to

$$\begin{aligned}\bar\psi_L\phi_L\,\bar\chi_L\lambda_L &= -\tfrac12\bar\psi_L\lambda_L\,\bar\chi_L\phi_L+\tfrac18\bar\psi_L\gamma^{ab}\lambda_L\,\bar\chi_L\gamma_{ab}\phi_L\,,\\ \bar\psi_L\phi_L\,\bar\chi_R\lambda_R &= -\tfrac12\bar\psi_L\gamma^a\lambda_R\,\bar\chi_R\gamma_a\phi_L\,.\end{aligned}\tag{A.64}$$

An extra minus sign w.r.t. (A.63) appears here because fermions λ and ϕ are interchanged. In 5 dimensions the Fierz equation is

$$\bar\psi^i\phi^j\bar\chi^k\lambda^\ell=-\tfrac14\bar\psi^i\lambda^\ell\bar\chi^k\phi^j-\tfrac14\bar\psi^i\gamma^a\lambda^\ell\bar\chi^k\gamma_a\phi^j+\tfrac12\bar\psi^i\gamma^{ab}\lambda^\ell\bar\chi^k\gamma_{ab}\phi^j.\tag{A.65}$$

In 6 dimensions ($\bar\psi_R=\bar\psi P_L$) one has

$$\begin{aligned}\bar\psi_R\phi_L\,\bar\chi_L\lambda_R &= -\tfrac14\bar\psi_R\gamma^a\lambda_R\,\bar\chi_L\gamma_a\phi_L+\tfrac{1}{48}\bar\psi_R\gamma^{abc}\lambda_R\,\bar\chi_L\gamma_{abc}\phi_L\,,\\ \bar\psi_R\phi_L\,\bar\chi_R\lambda_L &= -\tfrac14\bar\psi_R\lambda_L\,\bar\chi_R\phi_L+\tfrac18\bar\psi_R\gamma^{ab}\lambda_L\,\bar\chi_R\gamma_{ab}\phi_L\,.\end{aligned}\tag{A.66}$$

For those that want to go further, a more complicated identity for doublet spinors in 4 dimensions is

$$\varepsilon^{jk}\gamma^{ab}\lambda_i\bar\lambda_j\gamma_{ab}\lambda_k=8\varepsilon^{jk}\lambda_k\bar\lambda_i\lambda_j.\tag{A.67}$$

It uses the first equation of (A.64), symmetries of the bilinears and manipulations between the ε symbols.

A.4 Dimensional reduction of the spinors

To reduce 5-dimensional expressions to 4-dimensional ones, we identify γ_0 to γ_3 of 5 dimensions with those of 4 dimensions and we can take

$$\mathcal{C}_{(5)} = \mathcal{C}_{(4)}\gamma_5 , \tag{A.68}$$

where $\mathcal{C}_{(d)}$ is the charge conjugation matrix in d dimensions. Denoting for now the spinors of 5 dimensions with a tilde, we define the 4-dimensional chiral spinors as

$$\zeta^A = P_L\tilde{\zeta}^A . \tag{A.69}$$

Using the same choice of α in 4 as in 5 dimensions, we can then derive from the reality conditions that

$$\zeta_A = P_R\tilde{\zeta}^B\rho_{BA} = P_R\tilde{\zeta}_A , \tag{A.70}$$

where the lowering of index in 4 dimensions is determined by complex conjugation, while in 5 dimensions it is determined by ρ_{AB}.

For other spinors with i indices, we just replace ρ by ε.

B Groups and supergroups

For fixing my notations on algebras, I recapitulate the list of simple algebras[32] and their real forms in table 14. The conventions which I use for groups is that $\mathrm{Sp}(2n)=\mathrm{Sp}(2n,\mathbb{R})$ (always even entry), and $\mathrm{USp}(2m,2n)=\mathrm{U}(m,n,\mathbb{H})$. $\mathrm{S}\ell(n)$ is $\mathrm{S}\ell(n,\mathbb{R})$. Further, $\mathrm{SU}^*(2n) = \mathrm{S}\ell(n,\mathbb{H})$ and $\mathrm{SO}^*(2n) = \mathrm{O}(n,\mathbb{H})$. Note that in these bosonic algebras there are the following isomorphisms, some of which will be important later[33]

$$\begin{aligned}
\mathrm{SO}(3) &= \mathrm{SU}(2)=\mathrm{SU}^*(2)\,, \qquad \mathrm{SO}(2,1)=\mathrm{S}\ell(2)=\mathrm{SU}(1,1)=\mathrm{Sp}(2)\,,\\
\mathrm{SO}(4) &= \mathrm{SU}(2)\times\mathrm{SU}(2)\,, \qquad \mathrm{SO}(3,1)=\mathrm{SU}(2,\mathbb{C})=\mathrm{Sp}(2,\mathbb{C})\,,\\
& \quad \mathrm{SO}(2,2)=\mathrm{S}\ell(2)\times\mathrm{S}\ell(2)\,, \qquad \mathrm{SO}^*(4)=\mathrm{SU}(1,1)\times\mathrm{SU}(2)\,,\\
\mathrm{SO}(5) &= \mathrm{USp}(4)\,, \qquad \mathrm{SO}(4,1)=\mathrm{USp}(2,2)\,, \qquad \mathrm{SO}(3,2)=\mathrm{Sp}(4)\,,\\
\mathrm{SO}(6) &= \mathrm{SU}(4)\,, \qquad \mathrm{SO}(5,1)=\mathrm{SU}^*(4)\,, \qquad \mathrm{SO}(4,2)=\mathrm{SU}(2,2)\\
& \quad \mathrm{SO}(3,3)=\mathrm{S}\ell(4)\,, \qquad \mathrm{SO}^*(6)=\mathrm{SU}(3,1)\,,\\
\mathrm{SO}^*(8) &= \mathrm{SO}(6,2)\,.
\end{aligned} \tag{B.1}$$

Lie superalgebras have been classified in [127]. I do not have the time to go through the full classification mechanism of course, but will consider the most important superalgebras, the ‘simple Lie superalgebras’, which have no non-trivial invariant subalgebra. However, one should know that in superalgebras there are more subtle issues, as e.g. not any semi-simple

[32] I use the notations with capital letters though in many mathematical works the algebras are denoted by small letters.

[33] Note that the equality sign is not correct for the groups. These isomorphism are for the algebras. Also the covering groups of the orthogonal groups are equal to the groups mentioned at the right hand sides.

Table 14: *Real forms of simple bosonic Lie algebras. The second number in the notation for the real forms of exceptional algebras is the character (the number of non-compact − the number of compact generators). An extra possibility that does not occur in this table are the algebras over* $\mathbb{C}$*, i.e.* $\mathrm{SU}(n,\mathbb{C})$*,* $\mathrm{SO}(n,\mathbb{C})$ *and* $\mathrm{Sp}(2n,\mathbb{C})$*, which have the double number of real generators as their maximal compact subgroups* $\mathrm{SU}(n)$*,* $\mathrm{SO}(n)$ *and* $\mathrm{USp}(2n)$*.*

Compact	Real Form	Maximal compact subalg.
$\mathrm{SU}(n)$	$\mathrm{SU}(p,n-p)$	$\mathrm{SU}(p)\times\mathrm{SU}(n-p)\times\mathrm{U}(1), \quad 1\le p<n$
$\mathrm{SU}(n)$	$\mathrm{S}\ell(n)$	$\mathrm{SO}(n)$
$\mathrm{SU}(2n)$	$\mathrm{SU}^*(2n)$	$\mathrm{USp}(2n)$
$\mathrm{SO}(n)$	$\mathrm{SO}(p,n-p)$	$\mathrm{SO}(p)\times\mathrm{SO}(n-p)$
$\mathrm{SO}(2n)$	$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$
$\mathrm{USp}(2n)$	$\mathrm{Sp}(2n)$	$\mathrm{U}(n)$
$\mathrm{USp}(2n)$	$\mathrm{USp}(2p,2n-2p)$	$\mathrm{USp}(2p)\times\mathrm{USp}(2n-2p)$
$G_{2,-14}$	$G_{2,-14}$	G_2
$G_{2,-14}$	$G_{2,2}$	$\mathrm{SU}(2)\times\mathrm{SU}(2)$
$F_{4,-52}$	$F_{4,-52}$	$F_{4,-52}$
$F_{4,-52}$	$F_{4,-20}$	$\mathrm{SO}(9)$
$F_{4,-52}$	$F_{4,4}$	$\mathrm{USp}(6)\times\mathrm{SU}(2)$
$E_{6,-78}$	$E_{6,-78}$	$E_{6,-78}$
$E_{6,-78}$	$E_{6,-26}$	$F_{4,-52}$
$E_{6,-78}$	$E_{6,-14}$	$\mathrm{SO}(10)\times\mathrm{SO}(2)$
$E_{6,-78}$	$E_{6,2}$	$\mathrm{SU}(6)\times\mathrm{SU}(2)$
$E_{6,-78}$	$E_{6,6}$	$\mathrm{USp}(8)$
$E_{7,-133}$	$E_{7,-133}$	$E_{7,-133}$
$E_{7,-133}$	$E_{7,-25}$	$E_{6,-78}\times\mathrm{SO}(2)$
$E_{7,-133}$	$E_{7,-5}$	$\mathrm{SO}(12)\times\mathrm{SU}(2)$
$E_{7,-133}$	$E_{7,7}$	$\mathrm{SU}(8)$
$E_{8,-248}$	$E_{8,-248}$	$E_{8,-248}$
$E_{8,-248}$	$E_{8,-24}$	$E_{7,-133}\times\mathrm{SU}(2)$
$E_{8,-248}$	$E_{8,8}$	$\mathrm{SO}(16)$

superalgebra is the direct sum of simple superalgebras. A good review is [128]. The fermionic generators of such superalgebras are in representations of the bosonic part. If that 'defining representation' of the bosonic algebra in the fermionic generators is completely reducible, the algebra is said to be 'of classical type'. The others are 'Cartan type superalgebras' $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$, which we will further neglect. For further reference, I give in table 15 the list of the real forms of superalgebras 'of classical type' [129, 130, 23]. In this table

Table 15: *Lie superalgebras of classical type. Note that the superalgebras* $\mathrm{SU}(m|m)$*, which are indicated in this table with bosonic subalgebra* $\mathrm{SU}(m) \oplus \mathrm{SU}(m)$ *are often indicated as* $PSU(m|m)$*, while then* $\mathrm{SU}(m|m)$ *refers to the (non-simple) algebra including the* $\mathrm{U}(1)$ *factor.*

Name	Range	Bosonic algebra	Defining repres.	Number of generators
$\mathrm{SU}(m\|n)$	$m \geq 2$ $m \neq n$ $m = n$	$\mathrm{SU}(m) \oplus \mathrm{SU}(n)$ $\oplus\, \mathrm{U}(1)$ no $\mathrm{U}(1)$	$(m,\bar{n})\oplus$ $(\bar{m},n)$	m^2+n^2-1, $2mn$ $2(m^2-1), 2m^2$
$\mathrm{S}\ell(m\|n)$		$\mathrm{S}\ell(m) \oplus \mathrm{S}\ell(n)$ $\oplus\, \mathrm{SO}(1,1)$		if $m \neq n$
$\mathrm{SU}(m-p,p\|n-q,q)$		$\mathrm{SU}(m-p,p) \oplus \mathrm{SU}(n-q,q)$ $\oplus\, \mathrm{U}(1)$		if $m \neq n$
$\mathrm{SU}^*(2m\|2n)$		$\mathrm{SU}^*(2m) \oplus \mathrm{SU}^*(2n)$ $\oplus\, \mathrm{SO}(1,1)$		if $m \neq n$
$\mathrm{S}\ell\,'(n\|n)$		$\mathrm{S}\ell(n,\mathbb{C})$		
$\mathrm{OSp}(m\|n)$	$m \geq 1$ $n = 2,4,..$	$\mathrm{SO}(m) \oplus \mathrm{Sp}(n)$	(m,n)	$\frac{1}{2}(m^2-m+$ $n^2+n), mn$
$\mathrm{OSp}(m-p,p\|n)$		$\mathrm{SO}(m-p,p) \oplus \mathrm{Sp}(n)$		n even
$\mathrm{OSp}(m^*\|n-q,q)$		$\mathrm{SO}^*(m) \oplus \mathrm{USp}(n-q,q)$		m,n,q even
$D(2,1,\alpha)$	$0<\alpha\leq 1$	$\mathrm{SO}(4) \oplus S\ell(2)$	$(2,2,2)$	9, 8
$D^p(2,1,\alpha)$		$\mathrm{SO}(4-p,p) \oplus \mathrm{S}\ell(2)$		$p=0,1,2$
$F(4)$		$\overline{\mathrm{SO}(7)} \oplus \mathrm{S}\ell(2)$	$(8,2)$	21, 16
$F^p(4)$		$\mathrm{SO}(7-p,p) \oplus \mathrm{S}\ell(2)$		$p=0,3$
$F^p(4)$		$\mathrm{SO}(7-p,p) \oplus \mathrm{SU}(2)$		$p=1,2$
$G(3)$		$G_2 \oplus \mathrm{S}\ell(2)$	$(7,2)$	14, 14
$G_p(3)$		$G_{2,p} \oplus \mathrm{S}\ell(2)$		$p=-14,2$
$P(m-1)$	$m \geq 3$	$S\ell(m)$	$(m\otimes m)$	m^2-1, m^2
$Q(m-1)$	$m \geq 3$	$SU(m)$	Adjoint	m^2-1, m^2-1
$Q(m-1)$		$S\ell(m)$		
$Q((m-1)^*)$		$\mathrm{SU}^*(m)$		
$UQ(p,m-1-p)$		$\mathrm{SU}(p,m-p)$		

'defining representation' gives the fermionic generators as a representation of the bosonic subalgebra. The 'number of generators' gives the numbers of (bosonic,fermionic) generators in the superalgebra. I mention first the algebra as an algebra over $\mathbb{C}$, and then give different real forms of these algebras. With this information, you can reconstruct all properties of these algebras, up to a few exceptions. The names which I use for the real forms is for some algebras different from those in the mathematical literature [129, 130, 23], and chosen such that it is most suggestive of its bosonic content. There are isomorphisms as $\mathrm{SU}(2|1) = \mathrm{OSp}(2^*|2,0)$, and $\mathrm{SU}(1,1|1) = \mathrm{S}\ell(2|1) = \mathrm{OSp}(2|2)$. In the algebra $D(2,1,\alpha)$ the three $\mathrm{S}\ell(2)$ factors of the bosonic group in the anticommutator of the fermionic generators appear with relative weights 1, α and $-1-\alpha$. The real forms contain respectively $\mathrm{SO}(4) = \mathrm{SU}(2)\times\mathrm{SU}(2)$, $\mathrm{SO}(3,1) = \mathrm{S}\ell(2,\mathbb{C})$ and $\mathrm{SO}(2,2) = \mathrm{S}\ell(2)\times\mathrm{S}\ell(2)$. In the first and last case α should be real, while $\alpha = 1 + \mathrm{i}a$ with real a for $p = 1$. In the limit $\alpha = 1$ one has the isomorphisms $D^p(2,1,1) = \mathrm{OSp}(4-p,p|2)$.

For the real forms of $\mathrm{SU}(m|m)$, the one-dimensional subalgebra of the bosonic algebra is not part of the irreducible algebra. Furthermore, in that case there are subalgebras obtained from projection of those mentioned here with only one factor $\mathrm{SU}(n)$, $\mathrm{S}\ell(n)$, $\mathrm{SU}^*(n)$ or $\mathrm{SU}(n-p,p)$ as bosonic algebra.

C Comparison of notations

C.1 Other $N = 2$, $d = 4$ papers

Unfortunately the normalization of F and various other functions vary in the $N = 2$ literature. In table 16, I compare notations between various articles on $N = 2$ in $d = 4$. The first column is the notation used here, and for most part also e.g. in [77, 117, 47]. The exception is the Ricci sign (see below) that has been chosen negative in these papers. The column 'original' refers to the articles in the beginning of the development of $N = 2$ superconformal tensor calculus and special geometry. These are e.g. [40, 37, 48, 131, 132, 46, 60, 65, 1, 56, 67, 78, 79, 115, 133, 17]. The third column is the one presently mostly used in the Italian groups. It is especially the one that can be found in the paper that contains all the matter-coupled Lagrangians [10], and e.g. also in [134, 108].

The first row gives the signs in the Misner-Thorne-Wheeler classification [120]. The first sign is the 'signature', determining whether the authors use mostly + or mostly − convention for the metric. It determines whether we have to change $g_{\mu\nu}$ to $-g_{\mu\nu}$ to compare the papers. To compare with such papers, one has also to introduce an i factor for any single γ-matrix (assuming one uses $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = +2g_{\mu\nu}$ which is true in all these papers, but not in all the mathematical literature).

The second sign ('Riemann sign') determines whether

$$R^\mu{}_{\nu\rho\sigma} = s_2\left(\partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \ldots\right) . \tag{C.1}$$

Usually $s_2 = \pm 1$. However, in some papers $s_2 = -\frac{1}{2}$ which is related to not including the factor $p!$ in the definition of components of p-forms. This factor 2 is indicated in the second line. You can see that in these papers for the same reason also the field strength of vectors $F_{\mu\nu}$ have a different normalization.

Table 16: *Comparison of notations*

Here	original	Italian	[62, 63]	[81]	[106]
$(+++)$	$(+--)$	$(---)$	$(--)$	$(-)$	$(+-+)$
$R_{\mu\nu\rho}{}^{\sigma}$	$-R_{\mu\nu\rho}{}^{\sigma}$	$-2R_{\mu\nu\rho}{}^{\sigma}$	$-2R_{\mu\nu\rho}{}^{\sigma}$		$-2R_{\mu\nu\rho}{}^{\sigma}$
R	$-R$	$2R$			
$g_{\alpha\bar\beta}$	$-g_{A\bar B}$	g_{ij^*}	g_{ij^*}	$g_{i\bar\jmath}$	g_{ij^*}
$\mathcal{K}$	$-K+2\log\alpha$	$\mathcal{K}$	G	K	$\mathcal{K}$
F	$-\frac{\mathrm{i}}{4}F$		$-\mathrm{i}F$	F	$-F$
X^I	αX^I	L^Λ	L^Λ	L^Λ	L^Λ
Z^I	Z^I	X^Λ	X^Λ	X^Λ	X^Λ
F_I	$-\frac{\mathrm{i}}{4\alpha}F_I$	M_Λ	$-\mathrm{i}F_\Lambda$	M_Λ	$-F_\Lambda$
N_{IJ}	$-\frac{1}{\alpha^2}N_{IJ}$	$-N_{\Lambda\Sigma}$	$-N_{\Lambda\Sigma}$	$2\,\mathrm{Im}\,F_{\Lambda\Sigma}$	$\mathrm{i}N_{\Lambda\Sigma}$
$\mathcal{N}_{IJ}$	$\frac{\mathrm{i}}{\alpha^2}\mathcal{N}_{IJ}$	$\mathcal{N}_{\Lambda\Sigma}$	$-\mathrm{i}\mathcal{N}_{\Lambda\Sigma}$	$\mathcal{N}_{\Lambda\Sigma}$	
$C_{\alpha\beta\gamma}$	$e^{-K}Q_{ABC}$	$\mathrm{i}C_{ijk}$	C_{ijk}	$\mathrm{i}C_{ijk}$	$-\mathrm{i}C_{ijk}$
$F^I_{\mu\nu}$	$\alpha F^I_{\mu\nu}$	$-2\mathcal{F}^\Lambda_{\mu\nu}$	$4F^\Lambda_{\mu\nu}$	$\sqrt{2}\mathcal{F}^\Lambda_{\mu\nu}$	
g	$-\alpha^{-1}g$				
ϵ	2ϵ	$\sqrt{2}\epsilon$			
ψ_μ	ψ_μ	$\sqrt{2}\psi_\mu$			
γ_μ	γ_μ	$\mathrm{i}\gamma_\mu$			
Ω^I	$\alpha\Omega^I$				
η_i	η_i				
ϕ_μ	$\frac12\phi_\mu$				
$f_\mu{}^a$	$\frac12 f_\mu{}^a$				
Λ_K	$\frac12\Lambda_K$				
$V_\mu{}^i{}_j$	$\frac12\mathcal{V}_\mu{}^i{}_j$				
χ^i	χ^i				
$T^-_{ab}\varepsilon^{ij}$	T^{-ij}_{ab}				
ε^{abcd}	$\mathrm{i}\varepsilon^{abcd}$				

The third sign, 'Einstein sign', is the sign in $8\pi T_{\mu\nu} = s_3 \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$, where T_{00} is always positive. But it is easier to understand the product of sign 2 and sign 3 as the sign in the definition

$$s_2 s_3 R_{\mu\nu} = R^{\rho}{}_{\nu\rho\mu} \,. \tag{C.2}$$

Some signs can be recognized in papers as the sign of kinetic energies of the scalars and graviton. Positive kinetic energies imply for the kinetic terms

$$\mathcal{L} = -s_1 \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + s_1 s_3 \frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{C.3}$$

where we assumed that R is $g^{\mu\nu}R_{\mu\nu}$. See, however, the factor of 2 in the definition of R in some papers.

For some papers we did not give 3 signs, as the Ricci tensor, or even the Riemann tensor do not appear for all. If only one sign it is the signatures sign, if two signs it is signature and Riemann sign.

In the following rows, one finds the expression given in the first column in the notation of the corresponding papers. The freedom of the real parameter α, indicated in the second column, can be repeated in all columns, but looks most useful in this case. The sign of $\omega_\mu{}^{ab}$ is relevant in expressions as covariant derivatives on fermions, where we have

$$\partial_\mu + s_2 \tfrac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}, \tag{C.4}$$

and on vectors

$$\partial_\mu V^a + s_2 \omega_\mu{}^{ab} V_b. \tag{C.5}$$

We assume here the usual definition of the corresponding curvature as in the first line of (A.2), and that this curvature is related by vierbeins to the curvature $R_{\mu\nu\rho}{}^{\sigma}$ without changing the place of indices. Under these conditions, this sign is the same as the second sign of the Misner-Thorne-Wheeler signs. The papers with a factor 2 in the second line of table 16, have also a factor 2 in the first line of (A.2) as this is due to the same difference in normalization of components of 2-forms. Therefore the identification of the sign in (C.4) and (C.5) with the second Misner-Thorne-Wheeler sign remains valid.

Note also that in the old papers an antisymmetrization $[ab]$ was $(ab - ba)$ without the factor as in (A.1).

The symplectic matrices compare as follows between the notations here (left hand side) and in the 'old' notation (right hand side):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} U & 2\alpha^2 Z \\ \frac{1}{2\alpha^2} W & V \end{pmatrix} . \tag{C.6}$$

C.2 Translations from/to form language

We define the tangent space components of a generic p–forms, ϕ_p, according to

$$\phi_p = \frac{1}{p!} \mathrm{d}x^{\mu_1} \cdots \mathrm{d}x^{\mu_p} \phi_{\mu_p \cdots \mu_1} \,, \tag{C.7}$$

where the wedge product between forms will always be understood. The following relations are easily derived,

$$\begin{aligned} \mathrm{d}x^{\mu_0}\ldots\mathrm{d}x^{\mu_{d-1}} &= -\epsilon^{\mu_0\cdots\mu_{d-1}}\mathrm{d}x^0\ldots\mathrm{d}x^{d-1} \\ &= \epsilon^{\mu_0\cdots\mu_{d-1}}\mathrm{d}^dx\,, \end{aligned} \qquad \begin{aligned} &(C.8) \\ &(C.9) \end{aligned}$$

when we define

$$\mathrm{d}^dx \equiv -\mathrm{d}x^0\ldots\mathrm{d}x^{d-1}\,. \tag{C.10}$$

Exercise C.1: Obtain for 3-forms H and G in 6 dimensions, obtain that

$$\int G\,H = \tfrac{1}{6}\int \mathrm{d}^6x\sqrt{g}G_{\mu\nu\rho}\tilde{H}^{\,\mu\nu\rho}\;. \tag{C.11}$$

Note that we consider the differentials as space-time derivatives commuting with the spinors.

The differential d works on the right: $A = A_\mu \mathrm{d}x^\mu$, then $\mathrm{d}A = \partial_\nu A_\mu \mathrm{d}x^\mu \mathrm{d}x^\nu$. This implies e.g. that

$$F = \mathrm{d}A \qquad \Rightarrow \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,. \tag{C.12}$$

Note that also here conventions differ between authors. E.g. in the 'Italian papers' [10, 134, 108], the factor $1/p!$ is not introduced in (C.7), with as consequence that $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. XXX or do not change order and $A = A_\mu \mathrm{d}x^\mu$ and $\mathrm{d}A = \partial_\nu A_\mu \mathrm{d}x^\nu \mathrm{d}x^\mu$.

D Identities of very special geometry

For simplicity of notation, all the relations in this appendix are given for vector multiplets, though they apply for vector-tensor multiplets. In the latter case, all indices I should be replaced by $\tilde{I}$. The vector multiplets are defined in terms of the symmetric real constant tensor $\mathcal{C}_{IJK}$. The independent scalars are ϕ^x, but many quantities are defined by functions $h^I(\phi)$, satisfying

$$\mathcal{C}_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1, \tag{D.1}$$

and

$$\begin{aligned} &h_I(\phi) \equiv \mathcal{C}_{IJK}h^J(\phi)h^K(\phi) = a_{IJ}h^J, \qquad a_{IJ} \equiv -2\mathcal{C}_{IJK}h^K + 3h_Ih_J, \\ &\Gamma_{IJK} \equiv -\sqrt{\tfrac{2}{3}}\left(\mathcal{C}_{IJK} - 9h^L\mathcal{C}_{L(IJ}h_{K)} + 9h_Ih_Jh_K\right), \quad R_{IJKL} = 2\Gamma_{KM[J}\Gamma_{I]L}{}^M, \end{aligned} \tag{D.2}$$

where here and below I-type indices are lowered or raised with a_{IJ} or its inverse, which we assume to exist.

Define (with$_{,x}$ an ordinary derivative with respect to ϕ^x)

$$h^I_x \equiv -\sqrt{\tfrac{3}{2}}h^I_{,x}(\phi), \tag{D.3}$$

which, due to the constraint (D.1) satisfies $h_I h^I_x = 0$, leading to

$$h_{Ix} \equiv a_{IJ} h^J_x = \sqrt{\tfrac{3}{2}} h_{I,x}(\phi). \tag{D.4}$$

We then also have

$$h^I h_{Ix} = 0, \qquad h_I h^I{}_x = 0. \tag{D.5}$$

These quantities define the metric on the scalar space, which is the pull-back of the metric a_{IJ} to the subspace defined by (D.1):

$$g_{xy} \equiv h^I_x h^J_y a_{IJ} = -2 h^I_x h^J_y \mathcal{C}_{IJK} h^K. \tag{D.6}$$

The above relations can be written in matrix form

$$\begin{pmatrix} h^I \\ h^I_x \end{pmatrix} a_{IJ} \begin{pmatrix} h^J & h^J_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{xy} \end{pmatrix}. \tag{D.7}$$

We can find the inverse of the first and third $(n+1)\times(n+1)$ matrices on the left-hand side (using $h^y_I \equiv g^{yx} h_{Ix}$)

$$\begin{pmatrix} h^I \\ h^I_x \end{pmatrix} \begin{pmatrix} h_I & h^y_I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta^y_x \end{pmatrix} \;\rightarrow\; \begin{pmatrix} h_I & h^x_I \end{pmatrix} \begin{pmatrix} h^J \\ h^J_x \end{pmatrix} = \delta^J_I. \tag{D.8}$$

Multiplying the latter equation with a_{JK} leads to

$$h_I h_J + h^x_I h_{Jx} = a_{IJ}. \tag{D.9}$$

Using the decomposition of the unity as in (D.8), we can write (with ‘;’ a covariant derivative including a connection $h_{Jx;y} = h_{Jx,y} - \Gamma^z_{xy} h_{Jz}$ such that $g_{xy;z} = 0$)

$$\begin{aligned}
h_{Ix;y} &= \delta^J_I h_{Jx;y} = \left(h_I h^J + h^z_I h^J_z\right) h_{Jx;y} = \sqrt{\tfrac{2}{3}}\left(h_I h^J_y h_{Jx} + T_{xyz} h^z_I\right) = \sqrt{\tfrac{2}{3}}\left(h_I g_{xy} + T_{xyz} h^z_I\right), \\
h^I{}_{x;y} &= -\sqrt{\tfrac{2}{3}}\left(h^I g_{xy} + T_{xyz} h^{Iz}\right), \\
T_{xyz} &\equiv \sqrt{\tfrac{3}{2}} h_{Jx;y} h^J_z = -\sqrt{\tfrac{3}{2}} h_{Jx} h^J_{z;y} = \mathcal{C}_{IJK} h^I_x h^J_y h^K_z, \\
&\Rightarrow \quad \Gamma^z_{xy} = h^{Iz} h_{Ix,y} - \sqrt{\tfrac{2}{3}} T_{xyw} g^{wz} = h^z_I h^I_{x,y} + \sqrt{\tfrac{2}{3}} T_{xyw} g^{wz}.
\end{aligned} \tag{D.10}$$

The tensor T_{xyz} is symmetric. Comparing (D.9) and (D.2), we obtain

$$h^x_I h_{Jx} = -2\mathcal{C}_{IJK} h^K + 2 h_I h_J, \tag{D.11}$$

whose covariant derivative with respect to ϕ^y leads to

$$T_{xyz} h^x_I h^z_J = \mathcal{C}_{IJL} h^L_y + h_{(I} h_{J)y}. \tag{D.12}$$

Multiplying with another h^y_K, using again the two expressions for a_{IJ}, leads to

$$T_{xyz} h^x_I h^y_J h^z_K = \mathcal{C}_{IJK} + \tfrac{3}{2} a_{(IJ} h_{K)} - \tfrac{5}{2} h_I h_J h_K. \tag{D.13}$$

The curvature is

$$K_{xyzu} = R_{IJKL} h_x^I h_y^J h_z^K h_u^L = \tfrac{4}{3}\left(g_{x[u}g_{z]y} + T_{x[u}{}^w T_{z]yw}\right). \tag{D.14}$$

Finally, after a straightforward calculation, we get

$$\begin{aligned} T_{xyz;u} &= \sqrt{\tfrac{3}{2}}[g_{(xy}g_{z)u} - 2T_{(xy}{}^w T_{z)uw}], \\ C^{IJK}{}_{,x} &= 2T_{uvw;x} h^{Iu} h^{Jv} h^{Kw}. \end{aligned} \tag{D.15}$$

This formula was found in [135], but the factor was corrected in [136].

The domain of the variables should be limited to $h^I(\phi) \neq 0$ and the metrics a_{IJ} and g_{xy} should be positive definite. Due to relation (D.7) the latter two conditions are equivalent.

E Rigid supersymmetry

For convenience, we here give the formulae for rigid supersymmetry with vector and hypermultiplets.

The Lagrangian for the vector multiplet.

$$\begin{aligned} \mathcal{L} = &-\mathrm{i}F_I D_a D^a \bar{X}^I + \tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}_{ab}^{-I}\mathcal{F}^{-abJ} + \tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I \not{D}\Omega^{iJ} \\ &-\tfrac{1}{8}\mathrm{i}F_{IJ}Y^{ijI}Y_{ij}^J + \tfrac{1}{8}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}_i^J\Omega_j^K \\ &-\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}_i^I\gamma^{ab}\mathcal{F}_{ab}^{-J}\Omega_j^K + \tfrac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}_i^I\Omega_\ell^J\bar{\Omega}_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{k\ell} \\ &+\tfrac{1}{2}\mathrm{i}gF_I f_{JK}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij} - \tfrac{1}{2}\mathrm{i}gF_{IJ}f_{KL}^I\bar{X}^K\bar{\Omega}_i^J\Omega_j^L\varepsilon^{ij} - \mathrm{i}g^2F_I f_{JK}^I f_{LM}^J\bar{X}^K\bar{X}^L X^M \\ &+\text{h.c.}. \end{aligned} \tag{E.1}$$

We started from a holomorphic prepotential $F(X)$. The vector multiplets are denoted by indices I. The F_I, $F_{IJ}, \ldots$ are the derivatives. The holomorphic scalars are thus X^I, and X_I are the complex conjugates. Ω_i^I are the chiral fermions (left-handed). Ω^{iI} is the right-handed component. Covariant derivatives are given in (4.72), and the $\mathcal{F}_{ab}^{-I}$ are the anti-selfdual field strengths.

For the gauging I assumed here that F is an invariant function as in (4.73), although one may allow that F transforms in a quadratic function with real coefficients [56, 57] as in (4.74).

Now we come to the hypermultiplets. The real scalars are denoted as q^X, with X running over $4n_H$ values. The fermions are then ζ^A (left-handed chiral) or complex conjugates ζ_A (right-handed). A runs over $2n_H$ values. The metric of the hyper-Kähler manifold spanned by the scalars is g_{XY}. The latter is generated by vielbeins and the (hermitian) metric in tangent space $d^A{}_B$:

$$g_{XY} = f_{XiA} d^A{}_B f_Y^{iB}. \tag{E.2}$$

The vielbeins should satisfy reality conditions that are determined by a symplectic metric ρ_{AB}. Its complex conjugate is ρ^{AB} such that

$$\rho^{AB}\rho_{BC} = -\delta_C^A. \tag{E.3}$$

The reality conditions are

$$\left(f_X^{iA}\right)^* = f_{XiA} = f_X^{jB}\varepsilon_{ji}\rho_{BA}. \tag{E.4}$$

On the other hand, f_{iA}^X are the inverse matrices of f_X^{iA} as $4n_H \times 4n_H$ matrices, see (4.26). But one has to be careful to raise indices on the vielbeins with the metric, see (4.95).

However, if one is interested in positive definite kinetic energies, one can put $d^A{}_B = \delta^A{}_B$ without loosing generality.

The vielbeins should be covariant constant as in (4.27), where $\Gamma_{ZY}^X(q)$ is the Levi-Civita connection of the metric, and this defines $\omega_{XA}{}^B$.

In general hyper-Kähler manifolds we have for the curvature tensor

$$R_{XYZ}{}^W \equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V = -\tfrac{1}{2}f_X^{Ai}\varepsilon_{ij}f_Y^{jB}f_W^{kC}f_{kD}^Z W_{ABC}{}^D. \tag{E.5}$$

The symmetric tensor W appears in the action.

The 2-form complex structures are

$$\vec{J}_{XY} = -\mathrm{i}f_X^{iA}\vec{\sigma}_i{}^j f_{YjB}d^B{}_A\,. \tag{E.6}$$

The action is in general

$$\begin{aligned}S_{\rm hyp} \;=\int \mathrm{d}^4x\; &\Big(-\tfrac{1}{2}g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y - \left(\bar\zeta_A\not{\mathcal{D}}\zeta^B d^A{}_B + \text{h.c.}\right) + \tfrac{1}{2}W_{BC}{}^{DA}d^E{}_A\bar\zeta_E\zeta_D\bar\zeta^B\zeta^C\\ &-g\left(2\bar X^I t_I{}^{BA}\bar\zeta_C\zeta_B d^C{}_A + 2\mathrm{i}f_X^{iA}k_I^X\bar\zeta_B\Omega^{jI}\varepsilon_{ij} + \text{h.c.}\right) - gP_{Ii}{}^kY^{ijI}\varepsilon_{jk}\\ &-2g^2\bar X^I X^J k_{(I}^X k_{J)}^Y g_{XY}\Big)\,.\end{aligned} \tag{E.7}$$

The covariant derivatives in (E.7) are given in (4.105).

These equations include already the coupling of vector and hypermultiplets. This coupling is determined by gauged isometries. These are determined by Killing vectors k_I^X of the manifold that are triholomorphic, which means that they are derivable from a moment map according to

$$\partial_X\vec{P}_I = -\tfrac{1}{2}\vec{J}_{XY}k_I^Y\,. \tag{E.8}$$

The fact that this has a solution is equivalent to the requirement (4.53). The matrix $t_I{}^{AB}$ occurring in the action is defined as

$$t_I{}^{AB} = \rho^{AC}t_{IC}{}^B, \qquad t_{IA}{}^B = \frac{1}{2}f_{iA}^Y\mathfrak{D}_Y k_I^X f_X^{iB}\,. \tag{E.9}$$

The remarks on the simple example given around (4.106) apply also here and lead again to (4.107).

The general result is the sum of (E.1) and (4.103), which still includes the auxiliary fields Y^{ijI}.

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